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CRYSTALLINE COHOMOLOGY AND GL(2, $\mathbb{Q}$)

BY

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ABSTRACT

This paper applies recent advances in crystalline cohomology to the classical case of open elliptic modular curves. In so doing control is gained over the action of inertia in the Galois representations attached to modular forms. Our aim is to study the modular Galois representations attached to automorphic forms mod p of weight $k \geq 2$. We generalize to higher weight k several results which were previously accessible only in the case of weight 2 where jacobian varieties can be invoked. Additionally we reconsider Gross's theorem on companion forms in a crystalline context.

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Introduction

In this paper we apply recent advances in crystalline cohomology to the classical case of elliptic modular forms. The main tool is Faltings's Comparison Theorem for p-adic étale and crystalline cohomology in the case of open varieties with smooth normal crossings compactifications ([7, Theorem 5.3]). This result is applied to open elliptic modular curves, the context of Faltings's earlier work [8]. Our aim is to study the modular Galois representations attached to automorphic forms mod p of weight $k \geq 2$. The restrictions on the results obtained are dictated by the restrictions in the crystalline theory. It is helpful to make these explicit at the outset:

I. The Comparison Theorem is only established in the case of good reduction, so we shall always require that p not divide the level. Otherwise for $k > 2$ the theory either does not exist or it is not as useful.

II. With no restrictions on the weight k there is a $\mathbb{Q}_p$-theory. However in this paper we are exclusively interested in modular representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ arising from automorphic forms and hence require a $\mathbb{Z}_p$-theory. For this one must impose the condition $k < p$. At the limiting case $k = p$ some remnants of the theory should remain, see [1].

We begin with the foundations. Fix a level N and a weight $k \geq 2$. Denote by $S_k(\Gamma_1(N)$ the space of cusp forms of weight k for $\Gamma_1(N)$ and by $\mathbb{T}$ the Hecke algebra acting on this space. We first show that for a maximal ideal $\mathbf{m} \subseteq \mathbb{T}$ of residue characteristic $p > k$ with $(p, N) = 1$ the modular Galois representation $\rho_{\mathbf{m}}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}(2, \overline{\mathbb{F}}_p)$ is crystalline with the weights 0 and $(k-1)$ each occurring with multiplicity one. We furthermore show that in this case the Comparison Theorem between p-adic étale and crystalline cohomology is Hecke-equivariant. The equivariance follows easily from functorial properties except in the case of T_p, so it is here that all our efforts are concentrated.

The first application of these foundations is the essentially immediate result of Multiplicity One for maximal ideals $\mathbf{m} \subset \mathbb{T}$ of residue characteristic $p > k$, $p \nmid N$, such that the associated modular Galois representation $\rho_{\mathbf{m}}$ is irreducible (Theorem 2.1). In case $\rho_{\mathbf{m}}$ is reducible crystalline methods show only that $\rho_{\mathbf{m}} = \alpha\chi^{k-1} \oplus \beta$ with α, β characters of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ unramified outside N and χ the p-cyclotomic character. We next classify these reducible representations $\rho_{\mathbf{m}}$ occurring in cusp forms of type (N, k). Suppose $\mathbf{m} \subseteq \mathbb{T}$ with $\rho_{\mathbf{m}}$ reducible is a *new* maximal ideal in the sense that $\rho_{\mathbf{m}}$ does not occur in cusp forms of any level properly dividing N. Then if $p > k+1$ we prove that $\mathbf{m}$ arises from an Eisenstein series of type (N, k), cf. Theorem 3.38. To prove these results, we first use geometric methods to construct Eisenstein series $E(\alpha, \beta)$ attached to two Dirichlet characters α and β. For a reducible $\rho_{\mathbf{m}}$ the Eichler–Shimura relations give the same T_ℓ-eigenvalues as an Eisenstien series $E(\alpha, \beta)$ and the crystalline theory shows that the T_p-eigenvalue is the same as that of $E(\alpha, \beta)$. So we have to analyze the possible U_ℓ-eigenvalues for $\ell | N$. This study of bad reduction occupies much of Section 3 and relies on both the Picard–Lefschetz theory for semi-stable reduction and the work of Katz–Mazur.

Having classified $\mathbf{m}$ with $\rho_{\mathbf{m}}$ reducible we turn our attention to Multiplicity One in the Eisenstein case in Section 4. We show in a special case that Multiplicity One holds in the Eisenstein case for higher weight k. The case we can treat is of prime level and is the analogous case in higher weight to that considered by Mazur [14] in the case of weight 2. It should be pointed out that Multiplicity One in the Eisenstein case may well be false in general; cf. the theorem of Kurihara [12].

Lastly we reconsider the theorem of Gross [9] on the existence of companion forms. Gross proved this theorem by reducing to weight 2 and there studying bad reduction. We work in weight k in a case of good reduction by crystalline methods. This proof is unlike those of the other results in the paper. Whereas our other applications require only the general results on the crystalline cohomology of modular curves found in Section 1, here one really has to compute with Frobenius in crystalline cohomology. We know of no other similar applications of crystalline methods to Shimura varieties. But the topic is certainly in the air. Coleman and Voloch [3] have another new proof of Gross's Theorem which like ours does not use Multiplicity One. Moreover their techniques yield results in the case $p = k$ whereas our crystalline methods do not.

1. Hecke operators in crystalline cohomology

To set notation, fix a level $N \geq 3$ and denote by $\Gamma(N) \subseteq \mathrm{SL}(2, \mathbb{Z})$ the subgroup of matrices which are congruent to the identity modulo N. The open modular curve $Y(N)$ corresponding to $\Gamma(N)$ classifies elliptic curves with full level N-structure. There is then a universal elliptic curve $\pi\colon E \longrightarrow Y = Y(N)$. By adding finitely many cusps to $Y(N)$ we obtain a complete curve $X(N)$. The universal elliptic curve E extends to a semi-abelian variety over $X(N)$, giving rise to the semi-stable compactification $\overline{\pi}\colon \overline{E} \longrightarrow X = X(N)$. Everything is defined over $\mathbb{Z}[1/N, e^{2\pi i/N}]$ or over the extension L of $\mathbb{Q}_p$ generated by the Nth roots of unity.

Fix a prime p, $(p, N) = 1$, and work over the field L. Set $\mathbb{V} = R^1\pi_{*,\text{ét}}(\mathbb{Z}_p)$. Then $\mathbb{V}$ is an étale sheaf on $Y \otimes L$. View the structure sheaf $\mathcal{O}$ as a crystalline sheaf on E and set $\mathcal{E} = R^1\pi_{*,\text{crys}}(\mathcal{O})$. Then in the terminology of [7] we have $\mathcal{E} \in \mathcal{MF}^{\nabla}_{[0,1]}(Y)$. As $\overline{\pi}$ is logarithmically smooth, the crystalline sheaf $\mathcal{E}$ is associated to the étale sheaf $\mathbb{V}$ by [7, Theorem 6.2]. Hence, functorially for an integer $k \geq 2$, $\operatorname{Symm}^{k-2}(\mathcal{E}) \in \mathcal{MF}^{\nabla}_{[0,k-2]}(Y)$ is associated to $\operatorname{Symm}^{k-2}(\mathbb{V})$. From [7, Theorem 5.3] it then follows that for $p > k$ $H^1_{\text{crys}}(Y, \operatorname{Symm}^{k-2}(\mathcal{E}))$ is the F-crystal corresponding to the dual of $H^1_{\text{ét}}(Y_{\overline{L}}, \operatorname{Symm}^{k-2}(\mathbb{V}))$. A similar statement holds for $H^1_!$ and $H^1_{\text{par}} = \operatorname{Image}(H^1_! \longrightarrow H^1)$.

The crystal $H^1_{\text{crys}}(Y, \operatorname{Symm}^{k-2}(\mathcal{E}))$ has weights 0 and $(k-1)$ by the arguments of [8]. Namely we show that in the de Rham complex

$$\operatorname{Symm}^{k-2}(\mathcal{E}) \longrightarrow \operatorname{Symm}^{k-2}(\mathcal{E}) \otimes \Omega^1(cusps)$$

the associated graded pieces gr^i_F are acyclic for $0 < i < k-1$. Let ω be the bundle of differentials on the universal elliptic curve, so $\omega = \overline{\pi}_*\Omega^1_{\overline{E}/X}$. The connection on $\mathcal{E}$ induces an $\mathcal{O}_X$-linear map

$$\operatorname{gr}^1_F(\mathcal{E}) = \omega \longrightarrow \operatorname{gr}^0_F(\mathcal{E}) \otimes \Omega^1(cusps) = \omega^{\otimes -1} \otimes \Omega^1(cusps).$$

This map is given by the Kodaira–Spencer class $\kappa\colon \omega^{\otimes 2} \xrightarrow{\cong} \Omega^1(cusps)$ and is thus itself an isomorphism. This is the assertion for $k = 3$; other k then follow similarly by linear algebra. Note that there is no problem with factorials $n!$'s etc. as long as $k - 2 < p$. Also

$$F^{k-1} \cong M_k(\Gamma(N)) = \Gamma(X, \omega_E^{\otimes(k-2)} \otimes \Omega^1(cusps)),$$

where $M_k(\Gamma(N))$ denotes modular forms for $\Gamma(N)$ of weight k. Furthermore

$$F^0/F^{k-1} \cong S_k(\Gamma(N))^* \cong H^1(X, \omega_E^{\otimes 2-k})$$

with $S_k(\Gamma(N)) = \Gamma(X, \omega_E^{\otimes(k-2)} \otimes \Omega^1)$ denoting the space of cusp forms of weight k for $\Gamma(N)$. Again similar statements hold for $H^1_!$ and $H^1_{\text{par}} = \operatorname{Image}(H^1_! \longrightarrow H^1)$. Of course for the filtration on $H^1_{\text{par}}(Y_{\overline{L}}, \operatorname{Symm}^{k-2}(\mathcal{E}))$ one has $F^{k-1} \cong S_k(\Gamma(N))$. Various dualities are respected.

Suppose $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$ is a congruence subgroup with $\Gamma \supseteq \Gamma(N)$. The open Riemann surface $Y(\Gamma)$ is the quotient $\mathbb{H}/\Gamma$ of the Poincaré upper half plane $\mathbb{H}$. Correspondingly $X(\Gamma) = \mathbb{H}/\Gamma \cup \{cusps\}$. The moduli space $Y(\Gamma)$ and its compactification $X(\Gamma)$ can be defined over a number field contained in $\mathbb{Q}(e^{2\pi i/N})$. Frequently the most natural model for $X(\Gamma)$ will be over $\mathbb{Q}(e^{2\pi i/N})$. Our primary example will be

$$\Gamma = \Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) | \, c \equiv 0 \bmod N, \, a \equiv d \equiv 1 \bmod N \right\}.$$

In this case $X(\Gamma) = X_1(N)$ and $Y(\Gamma) = Y_1(N)$ can be defined over $\mathbb{Q}$. However a better model is the one over $\mathbb{Q}(e^{2\pi i/N})$ associated to the "balanced" $\Gamma_1(N)$ moduli problem. In general let K be the completion of a field of definition of $X(\Gamma)$ at a prime above p. The p-adic sheaf $\mathrm{Symm}^{k-2}(\mathbb{V})$ on $Y(N)$ has a natural $\Gamma/\Gamma(N)$ action. A general foundational comment is required here. We want to view $Y(\Gamma)$ as the quotient $Y(N)/(\Gamma/\Gamma(N))$. Unfortunately, if $\Gamma/\Gamma(N)$ has fixed points on $Y(N)$ it is necessary to use either stacks or an auxiliary level M structure in order to do this. We will write proofs throughout for the case that Γ has no elliptic elements, omitting the routine modifications in case there are fixed points. The only technicality is that we will want to use Poincaré or Serre duality at various places. For this we need p prime to the order of the stabilizers of any fixed points. This can be accomplished, for example, either by assuming $p \geq 5$ or working with $\Gamma_1(N)$ for $N > 3$.

We obtain a sheaf on the quotient $Y(\Gamma) = Y(N)/(\Gamma/\Gamma(N))$ and if $k > 2$

$$H^1(Y(\Gamma)_{\overline{K}}, \mathrm{Symm}^{k-2}(\mathbb{V})) = H^1(Y(N)_{\overline{K}}, \mathrm{Symm}^{k-2}(\mathbb{V}))^{\Gamma/\Gamma(N)}.$$

Hence we can pass from results established for $X = X(N)$ and $Y = Y(N)$ to results for congruence subgroups $X(\Gamma)$ and $Y(\Gamma)$ with $\Gamma \supseteq \Gamma(N)$. We summarize as follows:

THEOREM 1.1: *Suppose $N \geq 3$ and $\Gamma \supseteq \Gamma(N)$ is a congruence subgroup. The moduli curves $Y(\Gamma)$ and $X(\Gamma)$ are defined over a number field contained in $\mathbb{Q}(e^{2\pi i/N})$; let K denote the completion of this number field at a prime above $p \nmid N$. Suppose $k > p$. Then the $\mathrm{Gal}(\overline{K}/K)$-representation*

$$V = H^1_{\mathrm{par}}(Y(\Gamma)_{\overline{K}}, \mathrm{Symm}^{k-2}(\mathbb{V}))$$

is crystalline. The F-crystal corresponding to the dual of V has a canonical

Frobenius filtration

$$M = F^0 \supset F^{k-1} \supset 0$$

with $F^{k-1} \cong S_k(\Gamma)$ *and* $F^0/F^{k-1} \cong S_k(\Gamma)^*$.

As we have seen, the crystal $H^1_{\text{crys}}(Y, \text{Symm}^{k-2}(\mathcal{E}))$ corresponds to the dual of the Gal$(\overline{L}/L)$-representation $H^1_{\text{ét}}(Y_{\overline{L}}, \text{Symm}^{k-2}(\mathbb{V}))$ for $p > k$. As usual similar statements hold for $H^1_!$ and $H^1_{\text{par}} = \text{Image}(H^1_! \longrightarrow H^1)$. We consider the case $Y = Y_1(N)/\mathbb{Q}$ and assert that this relation preserves Hecke operators.

The Hecke correspondences here are from X to itself. We now define these correspondences. Recall that the moduli space X classifies pairs (E, x) where E is a generalized elliptic curve and x: $\mathbb{Z}/N\mathbb{Z} \hookrightarrow E$ is a point of exact order N. For a prime r the correspondence $\mathcal{T}_r \subseteq X \times X$ is defined by

$$\text{pr}_2(\mathcal{T}_r \cdot ((E, x) \times X)) = \sum_{\varphi} (\varphi E, \varphi x),$$

where φ ranges over the $(r+1)$ isogenies of degree r with source E. If the prime r divides N then the correspondence $\mathcal{U}_r \subseteq X \times X$ is defined by

$$\text{pr}_2(\mathcal{U}_r \cdot ((E, x) \times X)) = \sum_{\varphi} (\varphi E, \varphi x),$$

where φ ranges over the r isogenies of degree r with source E whose kernels have trivial intersection with the subgroup generated by x. Additionally there are automorphisms $\langle a \rangle$, $(a, N) = 1$, of $X/\mathbb{Q}$ and w_ζ where ζ is a primitive N-th root of unity. The automorphism $\langle a \rangle$ is defined by

$$\langle a \rangle \colon (E, x) \mapsto (E, ax).$$

The automorphism w_ζ is defined by

$$w_\zeta(E, x) = (E/\langle x \rangle, x')$$

where x' is the point in $E/\langle x \rangle$ with the Weil pairing $\langle x', x \rangle = \zeta$. Finally there is a correspondence which we shall not need

$$\mathcal{U}'_r = w^*_\zeta \mathcal{U}_r \quad \text{for a prime } r|N.$$

The definition of $\mathcal{U}'_r$ is independent of the primitive N-th root of unity ζ. These Hecke correspondences are all defined over $\mathbb{Q}$, save for w_ζ which is defined over $\mathbb{Q}(\zeta)^+$.

All of the above Hecke correspondences induce correspondences on the universal elliptic curve $\pi\colon E \longrightarrow Y$ by extending by the universal isogenies. For example $\langle a\rangle$ acts on E by multiplication by a. Functorially this gives the action of $\langle a\rangle$ on $\mathbb{V} = R^1\pi_{*,\text{ét}}(\mathbb{Z}_\ell)$ as multiplication by a. The action of $\langle a\rangle$ on $\operatorname{Symm}^{k-2}(\mathbb{V})$ is therefore multiplication by a^{k-2}. Similarly all the Hecke correspondences act on the sheaves $\operatorname{Symm}^{k-2}(\mathbb{V})$.

Consider first $\mathcal{T}_\ell$ with $\ell \neq p$. The Hecke correspondence $\mathcal{T}_\ell$ has good reduction at p:

$$\begin{array}{ccc} & \mathcal{T}_\ell \subset X \times X & \\ {}^{\mathrm{pr}_1}\swarrow & & \searrow^{\mathrm{pr}_2} \\ X & & X, \end{array}$$

with pr_1 and pr_2 denoting the projection onto the first and second factor, respectively. By definition $T_\ell = \mathrm{pr}_{1,*} \circ \mathrm{pr}_2^*$. Hence the T_ℓ-equivariance follows from the known functoriality of the Comparison Theorem. Set $\mathbb{V}_p = \mathbb{V} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. It then also follows that the decomposition

$$H^1(Y_{\overline{L}}, \operatorname{Symm}^{k-2}(\mathbb{V}_p)) = H^1_{\mathrm{par}}(Y_{\overline{L}}, \operatorname{Symm}^{k-2}(\mathbb{V}_p)) \oplus H^1_{\mathrm{Eis}}(Y_{\overline{L}}, \operatorname{Symm}^{k-2}(\mathbb{V}_p)) \tag{1}$$

is preserved. Similarly the Hecke correspondence $\mathcal{U}_r$ has good reduction at p for a prime $r|N$, $r \neq p$. The induced Hecke operator U_r preserves the decomposition (1) and moreover U_r^{crys} is associated to $U_r^{\text{ét}}$.

The Hecke operator T_p is more interesting. There are étale $T_p^{\text{ét}}$ and de Rham T_p^{DR} defined on étale, respectively de Rham, cohomology in characteristic 0. There is also a crystalline T_p^{crys}, defined by

$$T_p^{\mathrm{crys}} = F_p + \langle p\rangle F_p^t,$$

where F_p denotes Frobenius and F_p^t is its adjoint with respect to the inner product. The operators $T_p^{\text{ét}}$ and T_p^{DR} are defined using the correspondence $\mathcal{T}_p$ as before.

THEOREM 1.2: *Assume $p > k$ and $p \nmid N$.*

(1) *T_ℓ^{crys} is associated to $T_\ell^{\text{ét}}$, for a prime $\ell \nmid pN$. U_r^{crys} is associated to $U_r^{\text{ét}}$ for a prime $r|N$, $r \neq p$.*

(2) *$T_p^{\mathrm{crys}} = T_p^{\mathrm{DR}}$, and this is associated to $T_p^{\text{ét}}$.*

Proof: First note that we can reduce immediately to the case of $\mathbb{Q}_p$-coefficients by considering the inclusion of the cohomology with $\mathbb{Z}_p$-coefficients. The point is that the hypotheses $p > k$ and $(p, N) = 1$ insure that the $\mathbb{Z}_p$-cohomology $H^1(Y_{\overline{L}}, \mathrm{Symm}^{k-2}(\mathbb{V}))$ is torsion-free and hence injects into the $\mathbb{Q}_p$-cohomology. Also assertion (1) is rather trivial and is treated in the comments above.

As for assertion (2), we will first establish the assertion on H^1_{par}, dealing with the Eisenstein part later. With $Y = Y_1(N)$, $(p, N) = 1$ we denote by $Z^\circ \longrightarrow Y$ the $(k-2)$-fold product of the universal elliptic curve E and $Z \longrightarrow X$ the desingularization (following Deligne [5]) of the corresponding product of Néron models. Then $Z - Z^\circ$ is a divisor with normal crossings. We have injections preserving scalar products:

$$\begin{array}{ccc} H^1_!(Y_{\overline{L}}, \mathrm{Symm}^{k-2}\mathbb{V}_p) & \hookrightarrow & H^{k-1}_!(Z^\circ_{\overline{L}}, \mathbb{Q}_p) \\ \downarrow & & \downarrow \\ H^1_{\mathrm{par}}(Y_{\overline{L}}, \mathrm{Symm}^{k-2}\mathbb{V}_p) & \hookrightarrow & H^{k-1}_{\mathrm{par}}(Z^\circ_{\overline{L}}, \mathbb{Q}_p) \\ \downarrow & & \downarrow \\ H^1(Y_{\overline{L}}, \mathrm{Symm}^{k-2}\mathbb{V}_p) & \hookrightarrow & H^{k-1}(Z^\circ_{\overline{L}}, \mathbb{Q}_p). \end{array}$$

There is a similar commutative diagram for crystalline cohomology; the Frobenius automorphisms correspond. Furthermore the Hecke correspondence $\mathcal{T}_p$ on $Y \times Y$ induces a correspondence $\mathcal{T}_p^\circ$ on $Z^\circ \times Z^\circ$ by extending $\mathcal{T}_p$ by the universal isogeny. Explicitly if φ is a p-isogeny the for the pair $(E, \varphi(E))$ we have $(x_1, \ldots, x_{k-2}) \in E^{k-2}$ mapped to $(\varphi(x_1), \ldots, \varphi(x_{k-2})) \in \varphi(E)^{k-2}$. The Eichler–Shimura relation provides a description of $\overline{\mathcal{T}}_p = \mathcal{T} \bmod p$. Letting $\mathcal{F}_p$ denote the Frobenius correspondence on $\overline{Y} = Y \times \mathbb{F}_p$ we have:

$$\overline{\mathcal{T}}_p = \mathcal{F}_p + \langle p \rangle \mathcal{F}_p^t. \tag{2}$$

This familiar relation is easily deduced. Observe that if $(E, x) \in \overline{Y}$ is an ordinary point then

$$\overline{\mathcal{T}}_p \cdot ((E, x) \times \overline{Y}) = (E/\mu_p, \overline{x}) + \sum_{C \subseteq E[p] \text{ étale}} (E/C, \overline{x}).$$

But if (E, x) is an ordinary point then $\mathrm{Frob}_p(E, x) = (E/\mu_p, \overline{x})$. Moreover if $C \subseteq E[p]$ is an étale subgroup then

$$\mathrm{Frob}_p(E/C, \overline{x}) = (E/E[p], \overline{x}) \cong (E, px). \tag{3}$$

So one easily finds that the sum $\sum_{C\subseteq E[p]\,\text{étale}}(E/C,\overline{x})$ of equation (3) represents $\langle p\rangle\mathcal{F}_p^t$. Also $\mathcal{F}_p^t\circ\mathcal{F}_p = p\cdot\langle p\rangle$. This shows that the Eichler–Shimura relation holds on the ordinary locus of $\overline{Y}$. So equation (2) is valid since the ordinary locus is dense on $\overline{Y}$. Let $\overline{Z}^\circ$ denote $Z^\circ\times\mathbb{F}_p$ and correspondingly $\overline{\mathcal{T}}_p^\circ\subseteq\overline{Z}^\circ\times\overline{Z}^\circ$ denote the reduction of $\mathcal{T}^\circ$ modulo p. If $\mathcal{F}_p^\circ$ denotes the Frobenius correspondence on $\overline{Z}^\circ$, then exactly as above one sees that

$$\mathcal{T}_p^\circ = \mathcal{F}_p^\circ + \langle p\rangle(\mathcal{F}_p^\circ)^t. \tag{4}$$

Taking the Zariski closure we obtain a correspondence on $Z\times Z$, which modulo p is given by the decomposition $\mathcal{F}_p + \langle p\rangle(\mathcal{F}_p)^t$, together with possible components supported at ∞.

Now for constant coefficients characteristic classes in étale, de Rham, and crystalline cohomology correspond, cf. [7, Theorems 5.6 and 8.1]. So the induced maps on $H_!^{k-1}(Z_{\overline{L}}^\circ,\mathbb{Q}_p)$ and $H^{k-1}(Z_{\overline{L}}^\circ,\mathbb{Q}_p)$ correspond as well. Also, if we compose with the map $H_!^{k-1}\longrightarrow H^{k-1}$, components at ∞ disappear.

Finally, we check using the definition of the pairing in terms of Poincaré Duality that the closure of $(\mathcal{F}_p)^t$ operates on $H^1_{\text{par}}(Y_{\overline{L}},\text{Symm}^{k-2}(\mathcal{E}))$ as F_p^t. In general, suppose V is a smooth and proper variety over $\mathbb{Q}_p$ of dimension d. The pairing $\langle\ ,\ \rangle$ on $H^d_{\text{crys}}(V,\mathbb{Q}_p)$ is defined by $\langle\alpha,\beta\rangle = \text{Tr}_V(\alpha\cup\beta)$. Suppose $W\subset V\times V$ is a correspondence with cohomology class $c_W\in H^{2d}_{\text{crys}}(V\times V,\mathbb{Q}_p)$. The transpose correspondence W^t of W has cohomology class c_{W^t}. Then we have

$$\begin{aligned}\langle W\cdot\alpha,\beta\rangle &= \text{Tr}_{V\times V}(c_W\cup(\alpha\times\beta)) = (-1)^d\text{Tr}_{V\times V}(c_{W^t}\cup(\beta\times\alpha))\\ &= (-1)^d\langle W^t\cdot\beta,\alpha\rangle = \langle\alpha, W^t\cdot\beta\rangle.\end{aligned}$$

Hence the correspondence W^t acts on $H^d_{\text{crys}}(V,\mathbb{Q}_p)$ as the adjoint of W with respect to the natural inner product $\langle\ ,\ \rangle$. Applying these considerations to $H^{k-1}(Z,\mathbb{Q}_p)$ then verifies the assertion. This then shows that on H^1_{par}

$$T_p^{\text{DR}} = T_p^{\text{crys}} = \mathcal{F}_p + \langle p\rangle\mathcal{F}_p^t.$$

Also $T_p^{\text{ét}} = T_p^{\text{DR}}$ by the de Rham Conjecture [7, Theorem 8.1] as the characteristic classes of $\mathcal{T}_p$ correspond. This settles the case for cusp forms.

For the rest we denote by H_E^* the "cohomological" cokernel of $H_!^*\longrightarrow H^*$, so that we have the exact sequence

$$H^0\longrightarrow H_E^0\longrightarrow H_!^1\longrightarrow H^1\longrightarrow H_E^1\longrightarrow H_!^2\longrightarrow H^2\longrightarrow 0.$$

In de Rham or crystalline cohomology H_E^* is represented by the complex

$$(\mathrm{Symm}^{k-2}\mathcal{E} \longrightarrow \mathrm{Symm}^{k-2}\mathcal{E} \otimes \Omega(cusps)) \otimes \mathcal{O}/I_\infty$$

supported at the cusps. However, over the ordinary locus (and especially near the cusps) X^{ord} has a canonical Frobenius-lift Φ sending E to E modulo its multiplicative subgroup of order p. Also $\mathcal{T}_p = \Phi + \langle p\rangle\Phi^t$. This easily implies that $T_p^{\mathrm{DR}} = T_p^{\mathrm{crys}} = F_p + \langle p\rangle F_p^t$ on H_E^*. Finally from the Hodge–Tate theory we know that $T_p^{\text{ét}}$ and T_p^{DR} induce the same endomorphism on $\mathrm{gr}_F(H_{\mathrm{DR}}^1)$. As H_E^1 has pure weight $k-1$, this implies the assertion. ∎

We make the following remarks concerning Theorem 1.2:

I. Even if N is not prime to p, the de Rham Conjecture for Z implies that $H^1(Y_{\overline{L}}, \mathrm{Symm}^{k-2}\mathbb{V})$ is a de Rham representation associated to H_{DR}^1. Furthermore the T_p's correspond. This is true for H_{par}^1 and for $\mathrm{gr}_F(H_{\mathrm{DR}}^1)$, implying the result for $H^1(Y_{\overline{L}}, \mathrm{Symm}^{k-2}\mathbb{V})$.

II. The fact that $T_p^{\text{ét}} = T_p^{\mathrm{DR}}$ can also be seen using $\mathrm{gr}_F(H_{\mathrm{DR}}^1)$ (as maps in $\mathcal{MF}$ are strict for the filtration). However the Eichler–Shimura relation $T_p^{\mathrm{DR}} = F_p + F_p^t$ is surprisingly difficult to prove. The simplest way seemed to be the clumsy reduction to Z used above. A more canonical proof would require a more elaborate theory of correspondences in crystalline cohomology and their relation to étale cohomology.

2. Modular representations arising from automorphic forms

Fix a level $N \geq 3$ and a weight $k \geq 2$. For a prime p denote by ϑ_p the étale sheaf $\mathrm{Symm}^{k-2}(\mathbb{V})$ on $Y_1(N)/\mathbb{Q}$. Define $\overline{\vartheta}_p = \vartheta_p/p\vartheta_p$. The curve $Y_1(N)/\mathbb{Q}$ admits correspondences T_ℓ, $\ell \not| N$; $\langle d\rangle$, $d \in (\mathbb{Z}/N\mathbb{Z})^\times$; and U_ℓ and U_ℓ', $\ell | N$, which extend to correspondences on $X_1(N)/\mathbb{Q}$. They induce endomorphisms of $H_{\mathrm{par}}^1(Y_1(N)_{\overline{\mathbb{Q}}}, \vartheta_p)$. Let $\mathbb{T}_0$ be the $\mathbb{Z}$-algebra of endomorphisms of $H_{\mathrm{par}}^1(Y_1(N)_{\overline{\mathbb{Q}}}, \vartheta_p)$ generated by T_ℓ, $\ell \nmid N$, and $\langle d\rangle$, $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. Let $\mathbb{T}$ be the $\mathbb{Z}$-algebra of endomorphisms of $H_{\mathrm{par}}^1(Y_1(N)_{\overline{\mathbb{Q}}}, \vartheta_p)$ generated by $\mathbb{T}_0$ together with U_ℓ for $\ell | N$. The rings $\mathbb{T}$ and $\mathbb{T}_0$ are independent of p. Visibly $\mathbb{T}_0$ is a subring of the commutative ring $\mathbb{T}$.

Suppose that $\mathbf{m} \subseteq \mathbb{T}$ is a maximal ideal with residue field $\mathbf{k} = \mathbf{k}(\mathbf{m}) = \mathbb{T}/\mathbf{m}$ of characteristic p. Write Frob_r for a Frobenius element corresponding to the prime r in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then there is a unique semi-simple representation (up to isomorphism)

$$\rho_{\mathbf{m}}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}(2, \mathbf{k})$$

such that $\rho_{\mathbf{m}}$ is unramified for all primes $r \not| pN$ and for such primes

$$\mathrm{Tr}\,(\rho_{\mathbf{m}}(\mathrm{Frob}_r)) = T_r \bmod \mathbf{m} \ \text{ and } \det(\rho_{\mathbf{m}}(\mathrm{Frob}_r)) = \langle r \rangle r^{k-1} \bmod \mathbf{m}.$$

This is essentially due to Deligne. See, for example, [15, Proposition 5.1] or [9. Proposition 11.1]for detailed proofs. Note that due to the Brauer–Nesbitt Theorem the representation $\rho_{\mathbf{m}}$ is determined up to isomorphism by the characteristic polynomials of Frobenius for a set of primes r of density 1. Hence if $\mathbf{m}$ and $\mathbf{m}'$ are two maximal ideals of $\mathbb{T}$ with $\mathbf{m}_0 = \mathbf{m} \cap \mathbb{T}_0 = \mathbf{m}' \cap \mathbb{T}_0$, then $\rho_{\mathbf{m}}$ is isomorphic to $\rho_{\mathbf{m}'}$. Hence we may equally well refer to this isomorphism class of representations of $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ as $\rho_{\mathbf{m}_0}$, viewing it as being associated to the maximal ideal $\mathbf{m}_0 \subseteq \mathbb{T}_0$.

As a preliminary to applying the Eichler–Shimura relations to $H^1_{\mathrm{par}}(Y_1(N)_{\overline{\mathbb{Q}}}, \vartheta_p)$, we must discuss the distinction between arithmetic and geometric Frobenius. Note that on $H^*_{\text{ét}}(Y_1(N) \times \overline{\mathbb{F}}_\ell, \vartheta_p)$ we can let Frobenius at ℓ act via $Y_1(N)/\mathbb{F}_\ell$ or via $\overline{\mathbb{F}}_\ell$. The action of Frobenius via $Y_1(N)/\mathbb{F}_\ell$ is the geometric Frobenius F_{geom} and the action of Frobenius via $\overline{\mathbb{F}}_\ell$ is the arithmetic Frobenius F_{arith}. The two are related — F_{geom} and F_{arith} operate as inverses on $H^*_{\text{ét}}(Y_1(N) \times \overline{\mathbb{F}}_\ell, \vartheta_p)$. The Eichler–Shimura relations give that

$$T_\ell = F_{\mathrm{geom}} + \langle \ell \rangle F^t_{\mathrm{geom}}$$

as operators on $H^*_{\text{ét}}(Y_1(N)_{\overline{\mathbb{F}}_\ell}, \ \vartheta_p)$. Now the action of $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ on $H^1_{\mathrm{par}}(Y_1(N)_{\overline{\mathbb{Q}}}\vartheta_p)$ is via the second factor, i.e., $\mathrm{Frob}_\ell = F_\ell$ acts as arithmetic Frobenius. Hence we must work with the dual $H^1_{\mathrm{par}}(Y_1(N)_{\overline{\mathbb{Q}}}, \vartheta_p)^\vee$ as a $\mathbb{T}[\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})]$-module to have the relation $T_\ell = F_\ell + \langle \ell \rangle F^t_\ell$ valid for $\ell \not| pN$. Equivalently we consider the $\mathrm{Gal}\,(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$-representations $\mathbb{D}(H^1_{\mathrm{crys}})$ for each ℓ prime to pN. Another advantage of these representations is that they tend to have positive weights. However, this use of the dual gives us a contravariant correspondence between crystalline and étale, allowing ample cause for confusion but also making it more interesting.

By a standard argument (cf. [14, Sect. 2.14]) the $\mathbb{T}/\mathbf{m}^n[\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})]$-module

$$H^1_{\mathrm{par}}(Y_1(N)_{\overline{\mathbb{Q}}}, \vartheta_p)/\mathbf{m}^n H^1_{\mathrm{par}}(Y_1(N)_{\overline{\mathbb{Q}}}, \vartheta_p), \quad n \geq 0,$$

has a decomposition series whose subquotients are constituents of the dual $\rho^\vee_{\mathbf{m}}$ of $\rho_{\mathbf{m}}$.

In case $\rho_{\mathbf{m}}$ is irreducible, the crystalline information of Theorem 1.1 readily yields the following theorem.

THEOREM 2.1: *Let $\mathbf{m} \subseteq \mathbb{T}$ be a maximal ideal with $\mathbf{k} = \mathbb{T}/\mathbf{m}$ of characteristic p. Suppose $\rho_{\mathbf{m}}$ is irreducible. If $p > k$, then:*

(1) *$H^1_{\mathrm{par}}(Y_1(N)_{\overline{\mathbb{Q}}}, \overline{\vartheta}_p)[\mathbf{m}]^\vee$ is isomorphic to the $\mathbf{k}[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module corresponding to $\rho_{\mathbf{m}}$. In particular $\dim_{\mathbf{k}} H^1_{\mathrm{par}}(Y_1(N)_{\overline{\mathbb{Q}}}, \overline{\vartheta}_p)[\mathbf{m}] = 2$.*

(2) *The local ring $\mathbb{T}_{\mathbf{m}}$ is Gorenstein.*

Proof: Firstly observe that $H^1_{\mathrm{par}}(Y_1(N)_{\overline{\mathbb{Q}}}, \overline{\vartheta}_p)[\mathbf{m}] \neq 0$ since $\mathbb{T}$ operates faithfully on $H^1_{\mathrm{par}}(Y_1(N)_{\overline{\mathbb{Q}}}, \vartheta_p)^\vee$. This can be seen from the complex theory (specifically the Eichler–Shimura isomorphisms). Let W be the $\mathbf{k}[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module corresponding to $\rho_{\mathbf{m}}$. Then because $\rho_{\mathbf{m}}$ is irreducible we have that the semisimplification of $H^1_{\mathrm{par}}(Y_1(N)_{\overline{\mathbb{Q}}}, \overline{\vartheta}_p)[\mathbf{m}]^\vee$ is isomorphic to W^d for some $d \geq 1$ (cf. [14, Sect. 2.14]). By Theorem 1.1 the $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representation $H^1_{\mathrm{par}}(Y_1(N)_{\overline{\mathbb{Q}}_p}, \overline{\vartheta}_p)[\mathbf{m}]^\vee$ is crystalline with Hodge weights 0 and $k-1$; the same statement applies to $\rho_{\mathbf{m}}$.

Set $S_k = S_k(\Gamma_1(N))$. If M is the F-crystal corresponding to $(H^1_{\mathrm{par}}(Y_1(N)_{\overline{\mathbb{Q}}_p}, \overline{\vartheta}_p)[\mathbf{m}])^\vee$, then $\mathrm{gr}_F^{k-1}(M) \cong (S_k/pS_k)[\mathbf{m}]$ which is of dimension 1 over $\mathbf{k}$ by Multiplicity One. But taking the semisimplification we have the isomorphism

$$(H^1_{\mathrm{par}}(Y_1(N)_{\overline{\mathbb{Q}}}, \overline{\vartheta}_p)[\mathbf{m}]^\vee)^{ss} \cong W^d \quad \text{for some } d \geq 1$$

by Brauer–Nesbitt. As gr_F is exact (morphisms of F-crystals are strict), this forces $\rho_{\mathbf{m}}$ to correspond to an object in $\mathcal{MF}(\mathbb{Z}_p)$ with gr_F^{k-1} of dimension 1 over $\mathbf{k}$ and $d = 1$. Accordingly we must have that gr_F^0 is of dimension 1 over $\mathbf{k}$ since the total dimension is 2. In other words, both weights 0 and $k-1$ must occur in $\rho_{\mathbf{m}}$ with multiplicity one and hence $W = H^1_{\mathrm{par}}(Y_1(N)_{\overline{\mathbb{Q}}}, \overline{\vartheta}_p)[\mathbf{m}]^\vee$. This establishes (1). As $\mathrm{gr}_F^0(M) \cong (S_k^*/pS_k^*)[\mathbf{m}]$ then has dimension 1 over $\mathbf{k}$, this gives the Gorenstein condition in (2). ∎

In case $\rho_{\mathbf{m}}$ is reducible it is much harder to analyze. To apply the crystalline results we first observe the following:

LEMMA 2.2: *Let ψ: $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow \overline{\mathbb{F}}_p^\times$ be a character and denote by χ: $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathbb{F}_p^\times \subseteq \overline{\mathbb{F}}_p^\times$ the p-cyclotomic character. Denote by $I \subseteq \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ the inertia subgroup. Then ψ is crystalline of weight s, $0 \leq s \leq p-2$, if and only if $\psi = \chi^s$ on I.*

Proof: The statement depends only on inertia, so we pass to $\mathbb{Q}_p^{\mathrm{unr}}$. Let M be the F-crystal corresponding to the crystalline character ψ: $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{unr}}) \longrightarrow \overline{\mathbb{F}}_p^\times$ which we assume to be of weight s, $0 \leq s \leq p-2$. Then $M = F^s M = \overline{\mathbb{F}}_p \cdot e$ for a basis element e. We have that $\varphi^s e = ue$ for a unit u. The unit u determines the isomorphism class of the one-dimensional M of weight s subject to the change of variables that u and $u\lambda^\sigma\lambda^{-1}$ define isomorphic M for Frobenius automorphism σ and any unit λ in $\overline{\mathbb{F}}_p$. But then by Hilbert 90 $H^1(\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p), \overline{\mathbb{F}}_p^\times) = 1$. Hence up to isomorphism there is only such M; obviously χ^s is crystalline of weight s. ∎

The crystalline results in the reducible case then yield the following.

THEOREM 2.3: *Suppose $\rho_{\mathbf{m}}$ is reducible. Then*

$$\rho_{\mathbf{m}} = \alpha \oplus \beta\chi^{k-1}$$

where α, β: $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathbf{k}^\times$ are characters and χ: $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathbb{F}_p^\times \subseteq \mathbf{k}^\times$ is the p-cyclotomic character. The characters α and β are unramified outside pN. If $p > k$, then α and β are unramified at p.

Proof: If $p > k$ then at p $\rho_{\mathbf{m}}$ is crystalline, of weights 0 and $k-1$. As in the proof of 2.1, Multiplicity One on cusp forms implies that $\rho_{\mathbf{m}}$ corresponds to an object in $\mathcal{MF}(\mathbb{Z}_p)$ with gr_F^{k-1} and gr_F^0 both of dimension 1 over $\mathbf{k}$. The result then follows from 2.2. ∎

3. Classification of $\mathbf{m} \subseteq \mathbb{T}$ with $\rho_{\mathbf{m}}$ reducible

3.1 CONSTRUCTING EISENSTEIN SERIES VIA GEOMETRY. Before studying Eisenstein ideals we must first review the relevant basic material on the modular curve $X_1(N)$ and the classical function theory associated to $\Gamma_1(N)$.

Firstly recall that the cusps of $X_1(N)$ correspond to degenerate elliptic curves and may be parametrized by pairs (ζ, α) with $\zeta \in \mu_N$, $\alpha = i/N_e \in \mathbb{Q}/\mathbb{Z}$ with $N_e = N_e(\zeta, \alpha)$ equal to the denominator of α. Define $N_m = N_m(\zeta, \alpha)$ by $N = N_m N_e$. The parameters (ζ, α) correspond to the data $(E = \mathbb{G}_m/q^{\mathbb{Z}}, x = \zeta \cdot q^\alpha)$, which is defined over $\mathbb{Z}[1/N, \zeta][[q^{1/N_e}]]$. Note that the base $\mathbb{Z}[1/N, \zeta][[q^{1/N_e}]]$ has automorphisms $q^{1/N_e} \mapsto \rho q^{1/N_e}$, $\rho^{N_e} = 1$. Therefore we have the equivalence:

$$(\zeta, \alpha) \sim (\zeta\rho, \alpha) \quad \text{for all } \rho \in \mu_{N_e}. \tag{5}$$

Denote by $\langle\zeta,\alpha\rangle$ the equivalence class of the parametrized cusp (ζ,α) with respect to the equivalence (5). We call $\langle\zeta,\alpha\rangle$ an *oriented cusp* of $X_1(N)$. Additionally the Tate curve $E=\mathbb{G}_m/q^{\mathbb{Z}}$ has automorphism group $\langle\pm 1\rangle$, inducing an equivalence on the set of parametrized cusps

$$(6)\qquad (\zeta,\alpha)\sim(\zeta^{-1},-\alpha).$$

The equivalence (6) induces an equivalence

$$(7)\qquad \langle\zeta,\alpha\rangle\sim\langle\zeta^{-1},-\alpha\rangle$$

on the set of oriented cusps. Denote by $[\zeta,\alpha]$ the equivalence class of an oriented cusp $\langle\zeta,\alpha\rangle$ with respect to the equivalence (7). Then $\{[\zeta,\alpha]\}$ is naturally identified with the set of cusps on $X_1(N)$. For q-expansions however we shall need parametrized cusps.

Cusps with $N_m=N$, $N_e=1$ are called *multiplicative cusps.* The oriented cusp $\infty=\langle\zeta_0,1\rangle$ where ζ_0 is a fixed N^{th} root of 1 is a multiplicative oriented cusp. The set of all multiplicative oriented cusps is then given by $\langle d\rangle\cdot\infty$, $d\in(\mathbb{Z}/N\mathbb{Z})^\times$. Cusps with $N_e=N$, $N_m=1$ are called *étale cusps.* The oriented cusp $0=(1,1/N)$ is an étale oriented cusp; the set of all étale oriented cusps is then given by $\langle d\rangle\cdot 0$, $d\in(\mathbb{Z}/N\mathbb{Z})^\times$. The remaining cusps with $N_m\neq 1$, $N_e\neq 1$ are called *cusps of mixed type.*

We now turn to function theory. Let R be a commutative ring with $1/N\in R$. Denote by $M_k(R)$ the R-module of modular forms of weight k for $\Gamma_1(N)$ defined over R. By definition, $M_k(R)=H^0(X_1(N)_{/R},\Omega^1_{X_1(N)}(cusps)\otimes\omega^{\otimes(k-2)})$. The cusp forms of weight k, $S_k(R)\subseteq M_k(R)$, are defined by $S_k(R)=H^0(X_1(N)_{/R},\Omega^1_{X_1(N)}\otimes\omega^{\otimes(k-2)})$. As in our earlier general foundational comment in Section 1, if $N\leq 3$ it is necessary to use stacks or auxiliary level structure for such a definition of $S_k(R)$. The Kodaira–Spencer class gives an isomorphism of sheaves $i\colon\omega^{\otimes 2}\xrightarrow{\cong}\Omega^1_{X_1(N)}(cusps)$ on $X_1(N)$, which in turn gives an isomorphism of R-modules $M_k(R)\cong H^0(X_1(N),\omega^{\otimes k})$. Via this isomorphism we define the notion of q-expansion of a modular form. For any parametrized cusp (ζ,α), $f\in M_k(R)$ has a q-expansion at (ζ,α), $f(\zeta,\alpha)(q)\in R[\zeta][[q^{1/N_e}]]$, defined by

$$f(\mathbb{G}_m/q^{\mathbb{Z}},x=\zeta\cdot q^\alpha)=f(\zeta,\alpha)(q)\cdot(dt/t)^{\otimes k}.$$

The customary and convenient notation $f(\zeta,\alpha)(q)$ for q-expansions is misleading in that $f(\zeta,\alpha)(q)$ is a power series in q^{1/N_e} and not q. We will occasionally abuse notation and write $f(\zeta,\alpha) = f(\zeta,\alpha)(q^{1/N_e})$ for the power series in q^{1/N_e} with coefficients in $R[\zeta]$ given by $f(\zeta,\alpha)(q)$. It is important to note that the q-expansion at (ζ,α) depends on the parameters and is not invariant under the equivalences (5) and (6). Specifically suppose $\sigma^{N_e} = 1$ and define $(q')^{1/N_e} = \sigma q^{1/N_e}$. Then

$$\left(\mathbb{G}_m/q^{\mathbb{Z}},\, \zeta\cdot q^{\alpha}\right) = \left(\mathbb{G}_m/(q')^{\mathbb{Z}},\, \zeta\sigma^{-N_e\alpha}\cdot (q')^{\alpha}\right).$$

Therefore

$$f(\zeta,\alpha)(q^{1/N_e}) = f(\zeta\sigma^{-N_e\alpha},\alpha)(\sigma q^{1/N_e}) \quad \text{for } \sigma\in\mu_{N_e}. \tag{8}$$

Likewise one has

$$f(\zeta^{-1},-\alpha)(q) = (-1)^k f(\zeta,\alpha)(q). \tag{9}$$

Remark 3.1: A modular form $f\in M_k(\Gamma_1(N))$ then has a q-expansion at a parametrized cusp (ζ,α) given as $f(\zeta,\alpha)(q) = \sum_{n=0}^{\infty} a_n(\zeta,\alpha)q^{n/N_e([\zeta,\alpha])}$. There are important cases when the coefficients $a_n(\zeta,\alpha)$ depend only on the oriented cusp $\langle\zeta,\alpha\rangle$, i.e., are invariant under the equivalence (5):

1. We see from equation (8) that the constant term $a_0(f;(\zeta,\alpha))$ is independent of the the parameters (ζ,α) used for the oriented cusp $c = \langle\zeta,\alpha\rangle$. Hence we write $a_0(f;c)$.
2. If the oriented cusp c is multiplicative then it has a unique parametrization $c = \langle\zeta,0\rangle$ for ζ some primitive N^{th} root of 1. Hence we write $f(c)(q) = \sum_{n=0}^{\infty} a_n(f;c)q^n$ in this case. In particular any $f\in M_k(\Gamma_1(N))$ has a well-defined q-expansion at the oriented cusp ∞.

So for $f\in M_k(\Gamma_1(N)$ we define:

$$a_0(f) = \sum_{\substack{\text{oriented}\\ \text{cusps } c}} a_0(f;c)c \in R[\textit{oriented cusps}]. \tag{10}$$

Note that there is a homomorphism of R-modules

$$\overline{a}_0\colon M_k(R)\longrightarrow H^0(\textit{cusps}, \omega^{\otimes k}\otimes R).$$

The cusp forms $S_k(R)\subseteq M_k(R)$ are then equal to the kernel of $\overline{a}_0$. Let $E_k(\mathbf{C})\subseteq M_k(\mathbf{C})$ be the space of Eisenstein series, i.e., the orthogonal complement to $S_k(\mathbf{C})$ under the Petersson inner product. Then, for $k\geq 3$, $\overline{a}_0\colon E_k(\mathbf{C})\longrightarrow$

$H^0(cusps, \omega^{\otimes k} \otimes \mathbb{C})$ is an isomorphism. (For $k = 2$ $\overline{a}_0$ is injective with 1-dimensional cokernel.) The relationship between a_0 and $\overline{a}_0$ is as follows. Observe that the differential $(dt/t)^{\otimes k}$ as a chosen generator canonically identifies

$$H^0(\mathit{oriented\ cusps}, \omega^{\otimes k} \otimes R) \cong R[\mathit{oriented\ cusps}].$$

On the other hand, there is only a *noncanonical* isomorphism between $H^0(cusps, \omega^{\otimes k} \otimes R)$ and $R[cusps]$. In fact, to choose an isomorphism is equivalent to choosing an oriented cusp defining each cusp. Therefore we have the commutative diagram below, where the vertical map is taking the quotient by $\langle \pm 1 \rangle$:

$$\begin{array}{ccc} & & H^0(\mathit{oriented\ cusps}, \omega^{\otimes k} \otimes R) \cong R[\mathit{oriented\ cusps}] \\ & \overset{a_0}{\nearrow} & \big\downarrow /\langle \pm 1 \rangle \\ M_k(R) & \xrightarrow{\ \overline{a}_0\ } & H^0(cusps, \omega^{\otimes k} \otimes R) \end{array}$$

We recall from Section 1 that we have the following Hecke correspondences on the universal elliptic curve $\overline{\pi}$: $\overline{E} \longrightarrow X_1(N)$:

- $Corr(T_\ell)$ with the prime ℓ not dividing N
- $Corr(U_\ell)$ if the prime $\ell | N$
- $Corr(\langle d \rangle)$, $(d, N) = 1$
- $Corr(w_{\zeta_T})$, where $N = TS$ with $(T, S) = 1$ and ζ_T is a primitive T^{th} root of unity. If ζ is a primitive N^{th} root of unity we write w_ζ instead of w_{ζ_N}.

The isomorphism of sheaves $\omega^{\otimes 2} \xrightarrow{\cong} \Omega^1(cusps)$ on $X_1(N)$ given by the Kodaira–Spencer class induces an isomorphism between $H^0(X_1(N)_{/R}, \Omega^1_{X_1(N)}(cusps) \otimes \omega^{\otimes(k-2)})$ and $H^0(X_1(N)_{/R}, \omega^{\otimes k})$. The space of modular forms $M_k = M_k(R)$ may be identified with either of these cohomology groups. The Hecke correspondences *Corr* on the universal elliptic curve $\overline{\pi}$: $\overline{E} \longrightarrow X_1(N)$ induce pull-back morphisms $Corr^*$ on

$$H^0(X_1(N)_{/R}, \Omega^1_{X_1(N)}(cusps) \otimes \omega^{\otimes(k-2)}) \quad \text{and} \quad H^0(X_1(N)_{/R}, \omega^{\otimes k}).$$

(There are also induced push-forward morphisms which we do not consider.) However a crucial point is that the Kodaira–Spencer isomorphism i: $\omega^{\otimes 2} \xrightarrow{\cong} \Omega^1_{X_1(N)}(cusps)$ on $X_1(N)$ is *not* Hecke-equivariant. Specifically under an isogeny Kodaira–Spencer introduces a factor equal to the degree of the isogeny. Hence there are *two* contravariant actions of the Hecke operators on M_k. The action which gives the classical formulae is that of $H^0(X_1(N), \omega^{\otimes k-2} \otimes \Omega^1(cusps))$. This is also the group arising from the de Rham complex and crystalline cohomology. Hence it is necessary to take this action in order for the Comparison Theorem between p-adic étale and crystalline cohomology to be Hecke equivariant. We identify $M_k(R) = H^0(X_1(N)_{/R}, \Omega^1_{X_1(N)}(cusps) \otimes \omega^{\otimes(k-2)})$. The action of $Corr(T_\ell)^*$ for the prime $\ell \not| N$ on $f \in M_k(R)$ will be denoted simply $T_\ell f$. The pull-back action $Corr(T_\ell)^*$ on $H^0(cusps, \omega^{\otimes k} \otimes R)$ will be denoted simply by T_ℓ. Accordingly we take the dual action on the cusps—thus cusps are pushed forward and functions on cusps (i.e., cohomology classes) are pulled back. For an oriented cusp $c \in R[oriented\ cusps] \cong H^0(oriented\ cusps, \omega^{\otimes k} \otimes R)$ we denote $Corr(T_\ell)^*c$ simply by $T_\ell c$. The pull-back actions for the rest of the Hecke operators will be similarly denoted. With this Hecke action on the cusps the homomorphism of R-modules

$$\overline{a}_0 \colon M_k(R) \longrightarrow H^0(cusps, \omega^{\otimes k} \otimes R).$$

is *not* Hecke-equivariant.

Suppose $N = ST$ with $(S, T) = 1$. Then there is a corresponding factorization of cusp data. Namely for any elliptic curve E we have

$$E[N] \xrightarrow{\cong} E[S] \times E[T] \text{ via } E[N] \ni P \mapsto (TP, SP). \tag{11}$$

In particular consider the Tate curve $\mathbb{G}_m/q^{\mathbb{Z}}$ with the point $c = \zeta_N^j q^{i/N_e}$ of exact order N where ζ_N is an N-th root of unity. Write $N_m(c) = N_m = S_m T_m$ and $N_e(c) = N_e = S_e T_e$ with $S = S_m S_e$, $T = T_m T_e$. Set $\zeta_S = \zeta_N^T$ and $\zeta_T = \zeta_N^S$. Under the isomorphism (11), this point of order N corresponds to the pair of points $(\zeta_S^j q^{iT_m/S_e}, \zeta_T^j q^{iS_m/T_e})$. Parametrized cusps $(\zeta_N^j, i/N_e)$ on $X_1(N)$ correspond one-to-one with pairs of parametrized cusps $((\zeta_S^{j(S)}, i(S)/S_e); (\zeta_T^{j(T)}, i(T)/T_e))$.

By simple computations on the Tate curve $\mathbb{G}_m/q^{\mathbb{Z}}$ we assemble a "formulaire" for the Hecke actions. A remark on the notation used in the formulae follows the proposition.

Proposition 3.2: *For a modular form $f \in M_k(R)$ and a cusp $(\zeta, \alpha = i/N_e)$ we have:*

1. $(\langle d \rangle f)(\zeta, \alpha)(q) = f(\zeta^d, d\alpha)(q)$.
2. $(T_\ell f)(\zeta, \alpha)(q) = \ell^{k-1} f(\zeta^\ell, \alpha)(q^\ell) + \frac{1}{\ell} \sum_{\zeta_\ell^\ell = 1} f(\zeta \zeta_\ell^{-\ell\alpha}, \ell\alpha)(\zeta_\ell q^{1/\ell})$.
3. *If* $\ell | N_m$, $(U_\ell f)(\zeta, \alpha)(q) = \dfrac{1}{\ell} \displaystyle\sum_{\zeta_\ell^\ell = 1} f(\zeta \zeta_\ell^{-\ell\alpha}, \ell\alpha)(\zeta_\ell q^{1/\ell})$.

If $\ell \not| N_m$, $$(U_\ell f)(\zeta,\alpha)(q) = \ell^{k-1} f(\zeta^\ell,\alpha)(q^\ell) + \frac{1}{\ell} \sum_{\substack{\zeta_\ell^\ell=1 \\ \zeta_\ell^{iN_m} \neq \zeta^{N/\ell}}} f(\zeta\zeta_\ell^{-\ell\alpha},\ell\alpha)(\zeta_\ell q^{1/\ell}).$$

4. *If* $S_m \neq 1$ *and* $S_e \neq 1$,

$$(w_{\zeta_S} f)\left((\zeta_S^{j(S)},\frac{i(S)}{S_e});(\zeta_T^{j(T)},\frac{i(T)}{T_e})\right)(q) = \frac{S_m^k}{S} f\left((\zeta_S^{-1/i(S)},\frac{j(S)^{-1}}{S_m});(\zeta_T^{j(T)S_m},\frac{i(T)S_e}{T_e})\right)\left(\zeta_S^{\frac{S_m j(S)}{iS}} q^{S_m/S_e}\right).$$

If $S_m = 1$ *(so* $S_e = S$*)*,

$$(w_{\zeta_S} f)\left((1,\frac{i(S)}{S});(\zeta_T^{j(T)},\frac{i(T)}{T_e})\right)(q) = \frac{1}{S} f\left((\zeta_S^{-1/i(S)},0);(\zeta_T^{j(T)},\frac{i(T)}{T_e})\right)(q^{1/S}).$$

If $S_e = 1$ *(so* $S_m = S$*)*,

$$(w_{\zeta_S} f)\left((\zeta_S^{j(S)},0);(\zeta_T^{j(T)},\frac{i(T)}{T_e})\right) = S^{k-1} f\left((1,\frac{j(S)^{-1}}{S});(\zeta_T^{j(T)S},\frac{i(T)}{T_e})\right)(q^S).$$

Remark 3.3: We will explain in detail the notation used in Proposition 3.2(2), the other formulas being completely analogous. Choose ρ so that $\rho^{N_e} = \zeta_\ell$. Then $f(\zeta\zeta_\ell^{-\ell\alpha},\ell\alpha)(\zeta_\ell q^{1/\ell})$ means the power series $f(\zeta\rho^{-\ell N_e\alpha},\ell\alpha)(\rho q^{1/\ell N_e})$ in $\rho q^{1/\ell N_e}$. The notation is justified since the power series is independent of the choice of ρ. If we instead took $\tilde\rho = \rho\tau$ with $\tau^{N_e} = 1$, then

$$\begin{aligned} f(\zeta\tilde\rho^{-\ell N_e\alpha},\ell\alpha)(\tilde\rho q^{1/N_e\ell}) &= f(\zeta\rho^{-\ell N_e\alpha}\tau^{-\ell N_e\alpha},\ell\alpha)(\tau\rho q^{1/\ell N_e}) \\ &= f(\zeta\rho^{-\ell N_e\alpha},\ell\alpha)(\rho q^{1/\ell N_e}) \end{aligned}$$

by formula (8).

This notational convention is also used in the next proposition.

The Hecke action on $H^0(\textit{oriented cusps},\omega^{\otimes k}\otimes R) \cong R[\textit{oriented cusps}]$ is analogously given by:

PROPOSITION 3.4:
1. $\langle d\rangle\langle\zeta,\alpha\rangle = \langle\zeta^d,d\alpha\rangle$.
2. $T_\ell\langle\zeta,\alpha\rangle = \ell^k\langle\zeta^\ell,\alpha\rangle + \ell\langle\zeta,\ell\alpha\rangle$.

3. *If* $\ell | N_m$, $U_\ell\langle \zeta, \alpha\rangle = \sum_{\zeta_\ell^\ell = 1} \langle \zeta\zeta_\ell^{-\ell\alpha}, \ell\alpha\rangle$.

If $\ell \not| N_m$, $U_\ell\langle \zeta, \alpha\rangle = \ell^k\langle \zeta^\ell, \alpha\rangle + \sum_{\substack{\zeta_\ell^\ell = 1 \\ \zeta_\ell^{iN_m} \neq \zeta^{N/\ell}}} \langle \zeta\zeta_\ell^{-\ell\alpha}, \ell\alpha\rangle$.

4. *If* $S_m \neq 1$ *and* $S_e \neq 1$,

$$w_{\zeta_S}\left\langle (\zeta_S^{j(S)}, \frac{i(S)}{S_e}); (\zeta_T^{j(T)}, \frac{i(T)}{T_e})\right\rangle =$$
$$S_m^k \left\langle (\zeta_S^{-1/i(S)}, \frac{j(S)-1}{S_m}); (\zeta_T^{j(T)S_m}, \frac{i(T)S_e}{T_e})\right\rangle.$$

If $S_m = 1$ (*so* $S_e = S$),

$$w_{\zeta_S}\left\langle (1, \frac{i(S)}{S}); (\zeta_T^{j(T)}, \frac{i(T)}{T_e})\right\rangle = \left\langle (\zeta_S^{-1/i(S)}, 0); (\zeta_T^{j(T)}, \frac{i(T)}{T_e})\right\rangle.$$

If $S_e = 1$ (*so* $S_m = S$),

$$w_{\zeta_S}\left\langle (\zeta_S^{j(S)}, 0); (\zeta_T^{j(T)}, \frac{i(T)}{T_e})\right\rangle = S^k\left\langle (1, \frac{j(S)^{-1}}{S}); (\zeta_T^{j(T)S}, \frac{i(T)}{T_e})\right\rangle.$$

Comparing Propositions 3.2 and 3.4 we see the degree of the isogeny entering, causing a_0 not to be Hecke equivariant:

Proposition 3.5: *For* $f \in M_k(R)$ *and* c *an oriented cusp, we have:*

$$\begin{aligned} a_0(T_\ell f; c) &= (1/\ell)a_0(f; T_\ell c), \\ a_0(U_\ell f; c) &= (1/\ell)a_0(f; U_\ell c), \\ a_0(\langle d\rangle f; c) &= a_0(f; \langle d\rangle c), \\ a_0(w_{\zeta_S} f; c) &= (1/S)a_0(f; w_{\zeta_S} c). \end{aligned}$$

In particular apply Proposition 3.4 to $\infty = \langle \zeta_0, 0\rangle$ to deduce that:

$$\begin{aligned} &T_\ell(\langle d\rangle\infty) = \ell(1 + \langle \ell\rangle\ell^{k-1})\langle d\rangle\infty,\ \ell \not| N, \\ &U_\ell(\langle d\rangle\infty) = \langle d\rangle(\ell\infty + \textstyle\sum_{j=1}^{\ell-1}(\zeta_0, j/\ell)),\ \ell | N, \\ &w_{\zeta_0}(\langle d\rangle\infty) = (-1)^k\langle d^{-1}\rangle 0. \end{aligned} \tag{12}$$

Likewise Proposition 3.2 applied to ∞ yields the classical formulae for Hecke operators on Fourier expansions. If $f \in M_k(R)$ has q-expansion at ∞ given by $f(\infty)(q) = \sum_{n=0}^\infty a_n q^n$ and $(\langle \ell\rangle f)(\infty)(q) = \sum_{n=0}^\infty b_n q^n$, then

$$\begin{aligned} &(T_\ell f)(\infty)(q) = \textstyle\sum a_{n\ell}q^n + \ell^{k-1}\sum b_n q^{n\ell},\ (\ell, N) = 1, \\ &(U_\ell f)(\infty)(q) = \textstyle\sum a_{n\ell}q^n, \ell | N. \end{aligned} \tag{13}$$

The notion of twisting a modular form by a Dirichlet character admits a description in terms of moduli which we now recall. Suppose M is a positive integer which divides the level N. Let τ be a Dirichlet character of conductor M. Consider (E, x) with x an exact MN-division point on the elliptic curve E. Then $E' = E/\langle Nx\rangle$ possesses canonical M-division points, namely $N/M\overline{x}$ and $\overline{y} = \mathrm{Im}(y)$ with $\langle Nx, y\rangle = \zeta_0$ is a fixed primitive Mth root of 1. For $\lambda \in (\mathbb{Z}/M\mathbb{Z})$, set $E_\lambda = E'/\langle \overline{y} + \lambda\frac{N}{M}\overline{x}\rangle$, $\varphi_\lambda\colon E \longrightarrow E_\lambda$ the natural isogeny, and $x_\lambda = \varphi_\lambda(x)$ on E_λ.

Definition 3.6: Suppose f is a modular form of weight k on $\Gamma_1(N)$. Define the modular form $f^\tau = f \otimes \tau$ of weight k and level NM by

$$f^\tau(E,x) = (1/M^{k+1}) \sum_{\lambda\in\mathbb{Z}/M\mathbb{Z}}\ \sum_{\mu\in(\mathbb{Z}/M\mathbb{Z})^\times} \tau(\mu)\zeta_0^{-\mu\lambda}\varphi_\lambda^* f(E_\lambda, x_\lambda).$$

If f has character ε then f^τ has character $\varepsilon\tau^2$.

We compute the effect of twisting on q-expansions adhering to the above notation. So consider $(E,x) = (\mathbb{G}_m/q^{\mathbb{Z}}, \zeta = \zeta_{MN})$ with $x = h$ a point of exact order MN. The curve E' is then $E/\langle Nx\rangle$. The Mth power map gives the identifications:

$$E' \xrightarrow{\cong} \mathbb{G}_m/q^{M\mathbb{Z}} \quad \text{with } \overline{x} = \zeta_N,\ \frac{N}{M}\cdot\overline{x} = \zeta_M, \text{ and } \overline{y} = q.$$

The curve E_λ is correspondingly given by $\mathbb{G}_m/\langle q\zeta_M^\lambda\rangle$ with $\overline{x}_\lambda = \zeta_N$. Hence if $f(q) = f(\infty)(q) = \sum_{n=0}^\infty a_n q^n$ we have:

$$\begin{aligned}
(14)\qquad f^\tau(q) &= 1/M^{k+1} \sum_{\lambda\in\mathbb{Z}/M\mathbb{Z}}\ \sum_{\mu\in(\mathbb{Z}/M\mathbb{Z})^\times} \tau(\mu)\zeta_0^{-\mu\lambda} M^k f(q\zeta_0^\lambda)\\
&= 1/M \sum_\lambda\sum_\mu \tau(\mu)\zeta_0^{-\mu\lambda}\left(\sum_{n=0}^\infty a_n (q\zeta_0^\lambda)^n\right)\\
&= \sum_{n=0}^\infty a_n\cdot 1/M\left(\sum_\lambda\sum_\mu \tau(\mu)\zeta_0^{(n-\mu)\lambda}\right) q^n\\
&= \sum_{n=0}^\infty \tau(n) a_n q^n.
\end{aligned}$$

We therefore recover the classical definition of twisting via the above effect on q-expansions.

From Definition 3.6 the effect of the Hecke operators on twists of modular forms is easily computed:

PROPOSITION 3.7:

1. $T_\ell(f^\tau) = \tau(\ell)(T_\ell f)^\tau$ *for a prime* $\ell \nmid N$.
2. $U_\ell(f^\tau) = \tau(\ell)(U_\ell f)^\tau$ *for a prime* $\ell | N$.
3. $\langle d\rangle \cdot f^\tau = \tau(d)^2(\langle d\rangle \cdot f)^\tau$.

Finally in this preliminary section we consider Eisenstein series. Recall that $E_k(\mathbb{C})$ is the orthogal complement to the cusp forms $S_k(\mathbb{C}) \cong H^0(X_1(N), \omega^{\otimes(k-2)} \otimes \Omega^1)$ in the space of modular forms $M_k(\mathbb{C}) \cong H^0(X_1(N), \omega^{\otimes(k-2)} \otimes \Omega^1(cusps))$. For simplicity we assume $k \geq 3$ so that a_0: $E_k(\mathbf{C}) \xrightarrow{\cong} H^0(cusps, \omega^{\otimes k} \otimes \mathbf{C})$. So then Eisenstein series are identified with their constant terms. In particular we can use this both to construct Eisenstein series and to compute Hecke actions on Eisenstein series. Suppose ε is a Dirichlet character of conductor N with $\varepsilon(-1) = (-1)^k$. Let $\tilde{E}(\varepsilon, 1) \in E_k(\mathbf{C})$ be the Eisenstein series with

$$a_0(\tilde{E}(\varepsilon,1)) = \sum_{d\in(\mathbb{Z}/N\mathbb{Z})^\times} \varepsilon(d)\langle d\rangle \cdot 0.$$

By definition $\tilde{E}(\varepsilon, 1)$ is a modular form of weight k on $\Gamma_1(N)$. We remark that if $k = 2$ then the image of a_0 is the codimension 1 subspace consisting of elements of trace 0. In this case it is necessary to assume that the character ε is primitive to satisfy the condition that the sum of the residues must be 0.

For any modular form f of weight k with q-expansion at ∞ given by $f(\infty)(q) = \sum_{n=0}^\infty a_n q^n$ we have the associated Dirichlet series $L(f,s) = \sum_{n=1}^\infty a_n n^{-s}$.

PROPOSITION 3.8:

1. $\langle d\rangle(\tilde{E}(\varepsilon,1)) = \varepsilon(d)\tilde{E}(\varepsilon,1)$
2. $T_\ell(\tilde{E}(\varepsilon,1)) = (\varepsilon(\ell) + \ell^{k-1})\tilde{E}(\varepsilon,1)$ for all $\ell \nmid N$
3. $U_\ell(\tilde{E}(\varepsilon,1)) = \ell^{k-1}\tilde{E}(\varepsilon,1)$ for all $\ell | N$.

Hence $L(\tilde{E}(\varepsilon,1),s) = C(\varepsilon)L(s,\varepsilon)L(s+1-k,1)$ *for a nonzero constant* $C(\varepsilon)$.

Proof: Set $\tilde{E} = \tilde{E}(\varepsilon,1)$. Then by Proposition 3.5 we have for c an oriented cusp $a_0(\langle d\rangle\tilde{E}; c) = a_0(\tilde{E}; \langle d\rangle c) = 0$ unless c is étale. Since

$$a_0(\tilde{E}; \langle d\rangle\langle 1, d'/N\rangle) = a_0(\tilde{E}; \langle 1, dd'/N\rangle) = \varepsilon(dd') = a_0(\varepsilon(d)\tilde{E}; \langle 1, d'/N\rangle),$$

we conclude that $a_0(\langle d\rangle\tilde{E}) = a_0(\varepsilon(d)\tilde{E})$. Hence $\langle d\rangle\tilde{E} = \varepsilon(d)\tilde{E}$, establishing (1).

For (2), Proposition 3.5 gives that $a_0(T_\ell\tilde{E}; c) = (1/\ell)a_0(\tilde{E}; T_\ell c) = 0$ unless the oriented cusp c is étale. We furthermore have

$$\begin{aligned} a_0(T_\ell\tilde{E};\langle 1,d'/N\rangle) &= (1/\ell)a_0(\tilde{E};T_\ell\langle 1,d'/N\rangle) \\ &= (1/\ell)a_0(\tilde{E};\ell^k\langle 1,d'/N\rangle+\ell\langle 1,\ell d'/N\rangle) \quad \text{by Prop. 3.4(2)} \\ &= a_0((\ell^{k-1}+\langle\ell\rangle)\tilde{E};\langle 1,d'/N\rangle). \end{aligned}$$

Therefore we conclude that $a_0(T_\ell\tilde{E})=a_0((\ell^{k-1}+\varepsilon(\ell))\tilde{E})$, implying (2).

Likewise for (3) we have that $a_0(U_\ell\tilde{E};c) = (1/\ell)a_0(\tilde{E};U_\ell c) = 0$ unless the oriented cusp c is étale. Moreover

$$\begin{aligned} a_0(U_\ell\tilde{E};\langle 1,d'/N\rangle) =&(1/\ell)a_0(\tilde{E};U_\ell\langle 1,d'/N\rangle) \\ =&(1/\ell)a_0(\tilde{E};\ell^k\langle 1,d'/N\rangle+\sum_{\zeta_\ell\neq 1}\langle\zeta_\ell^{-\ell d'/N},\ell d'/N\rangle) \\ &\text{by Proposition 3.4(3)} \\ =&a_0((\ell^{k-1}\tilde{E};\langle 1,d'/N\rangle), \end{aligned}$$

proving (3) and concluding the proof of the proposition. ∎

Having defined $\tilde{E}(\varepsilon,1)$ by giving its constant term, we see from Proposition 3.8 that it is an eigenfunction for the weight k Hecke algebra $\mathbb{T}$ for $\Gamma_1(N)$. We need to determine its q-expansion at ∞, $\tilde{E}(\varepsilon,1)(\infty)(q)$. For this we use its functional equation. In general, suppose $f\in M_k(\Gamma_1(N))$ with $f(\infty)(q)=\sum_{n=0}^\infty a_nq^n$. Set $\check{f}=f-a_0$ and

$$D(f,s)=\frac{1}{i}\int_0^{i\infty}\check{f}(z)y^{s-1}dz=\int_0^\infty\check{f}(iy)y^{s-1}dy;$$

the integral is absolutely convergent for $\mathrm{Re}\,(s)>k$. If $\mathrm{Re}\,(s)>k$, then

$$D(f,s)=\Gamma(s)(2\pi)^{-s}L(f,s), \qquad \text{where } L(f,s)=\sum_{n=1}^\infty\frac{a_n}{n^s}.$$

Let

$$\gamma=\begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix}$$

and put $g=f\|[\gamma]_k\in M_k(\Gamma_1(N))$. So by definition $g(z)=N^{-k/2}z^{-k}f(-1/Nz)$ for z in the Poincaré upper half plane. Observe that $g=N^{-(k-2)/2}w_\zeta(f)$ where $\zeta=e^{2\pi i/N}$. We denote the q-expansion of g at ∞ by $g(\infty)(q)=\sum_{n=0}^\infty b_nq^n$ and set $\check{g}=g-b_0$. Then for $\mathrm{Re}\,(s)>k$,

$$(15)\quad D(f,s)=\int_0^{1/\sqrt{N}}\check{f}(iy)y^{s-1}dy+\int_{1/\sqrt{N}}^\infty\check{f}(iy)y^{s-1}dy$$

$$= \int_0^{1/\sqrt{N}} f(iy)y^{s-1}dy - N^{-s/2}(a_0/s) + \int_{1/\sqrt{N}}^{\infty} \check{f}(iy)y^{s-1}dy.$$

But using the functional equation $f(-1/Nz) = N^{k/2}z^k g(z)$ we have that

$$\begin{aligned} (16) \qquad \int_0^{1/\sqrt{N}} f(iy)y^{s-1}dy &= \int_{1/\sqrt{N}}^{\infty} f(i/Ny)N^{-s}y^{-1-s}dy \\ &= -i^k N^{-s/2}\left(\frac{b_0}{k-s}\right) + i^k N^{k/2-s}\int_{1/\sqrt{N}}^{\infty} \check{g}(iy)y^{k-1-s}dy. \end{aligned}$$

Now the integrals

$$\int_{1/\sqrt{N}}^{\infty} \check{f}(iy)y^{s-1}dy \quad \text{and} \quad \int_{1/\sqrt{N}}^{\infty} \check{g}(iy)y^{k-1-s}dy$$

are absolutely convergent for all s and so are holomorphic throughout the complex plane. Putting equations (15) and (16) together then, we see that $D(f,s)$ can be written as

$$\begin{aligned} (17) \qquad D(f,s) \;=\; & \int_{1/\sqrt{N}}^{\infty} \check{f}(iy)y^{s-1}dy + i^k N^{k/2-s}\int_{1/\sqrt{N}}^{\infty} \check{g}(iy)y^{k-1-s}dy \\ & -N^{-s/2}\left(\frac{a_0}{s} - i^k\frac{b_0}{s-k}\right). \end{aligned}$$

We record this information for future reference:

Proposition 3.9: *Let $f \in M_k(\Gamma_1(N))$ with $f(\infty)(q) = \sum_{n=0}^{\infty} a_n q^n$. For*

$$\gamma = \begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix},$$

set $g = f|[\gamma]_k$ and $g(\infty)(q) = \sum_{n=0}^{\infty} b_n q^n$. Then $D(f,s)$ has a meromorphic continuation throughout the complex plane with simple poles at $s=0$ and $s=k$. Moreover $\mathrm{Res}_{s=0} D(f,s) = -a_0$ and $\mathrm{Res}_{s=k} D(f,s) = i^k N^{-k/2} b_0$.

We now apply this proposition to find the q-expansion $\tilde{E}(1,\varepsilon)$. We shall need a formula (cf. [4, Chapter 9]) arising from the functional equation of $L(k,\varepsilon)$:

(18) For ε primitive with $\varepsilon(-1) = (-1)^k$,

$$L(k,\varepsilon) = \frac{\tau(\varepsilon)(2\pi)^k N^{-k} i^{-k}}{2\Gamma(k)} L(1-k,\varepsilon^{-1}).$$

Here $\tau(\varepsilon)$ denotes as usual the Gauss sum associated to ε.

PROPOSITION 3.10: *Suppose* $\varepsilon(-1) = (-1)^k$. *Set* $L(s,\varepsilon)L(s+1-k,1) = \sum_{n=1}^{\infty} a_n n^{-s}$. *Then*

$$\tilde{E}(\varepsilon,1)(\infty)(q) = C(\varepsilon)\sum_{n=1}^{\infty} a_n q^n, \quad \text{where } C(\varepsilon) = \frac{i^k}{(2\pi)^{-k}\Gamma(k)L(k,\varepsilon)}.$$

If ε *is primitive, then there is the equivalent formula*

$$C(\varepsilon) = \frac{(-1)^k 2N^k}{\tau(\varepsilon)L(1-k,\varepsilon^{-1})}.$$

Proof: Let $\tilde{f} = \tilde{E}(\varepsilon,1)$. By Proposition 3.8, $L(\tilde{f},s) = CL(s,\varepsilon)L(s+1-k,1)$ for some constant $C = C(\varepsilon)$. Then $D(\tilde{f},s) = C(2\pi)^{-s}\Gamma(s)L(s,\varepsilon)L(s+1-k,1)$. It follows that

$$\operatorname{Res}_{s=k} D(\tilde{f},s) = C(2\pi)^{-k}\Gamma(k)L(k,\varepsilon).$$

On the other hand, apply Proposition 3.9 to $\tilde{f} = \tilde{E}(\varepsilon,1)$. Then $g = f|[\gamma]_k = N^{-(k-2)/2}w_\zeta(f)$ satisfies

$$a_0(g;\infty) = N^{-(k-2)/2}a_0(w_\zeta(f);\infty) = N^{-(k-2)/2}N^{k-1}a_0(f;0) = N^{k/2},$$

and hence $\operatorname{Res}_{s=k} D(\tilde{f},s) = i^k$. Equate the two expressions for $\operatorname{Res}_{s=k} D(\tilde{f},s)$ and solve for $C = C(\varepsilon)$. We find

$$C = C(\varepsilon) = \frac{i^k}{(2\pi)^{-k}\Gamma(k)L(k,\varepsilon)}.$$

If ε is primitive, apply equation (18) to obtain the indicated formula for $C(\varepsilon)$ and conclude the proof of the proposition. ∎

Normalize by setting

$$E(\varepsilon,1) = \frac{1}{C(\varepsilon)}\tilde{E}(\varepsilon,1) = \frac{(-1)^k\tau(\varepsilon)L(1-k,\varepsilon^{-1})}{2N^k}\tilde{E}(\varepsilon,1).$$

Then

$$E(\varepsilon,1)(\infty)(q) = \sum_{n=1}^{\infty} a_n q^n \quad \text{where } L(s,\varepsilon)L(s+1-k,1) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{19}$$

and

$$a_0(E(\varepsilon,1)) = (-1)^k i^k (2\pi)^{-k}\Gamma(k)L(k,\varepsilon)\sum_{d\in(\mathbb{Z}/N\mathbb{Z})^\times} \varepsilon(d)\langle d\rangle\cdot 0. \tag{20}$$

If ε is primitive then by equation (18) this may be rewritten

$$a_0(E(\varepsilon,1)) = \frac{(-1)^k\tau(\varepsilon)L(1-k,\varepsilon^{-1})}{2N^k} \sum_{d\in(\mathbb{Z}/N\mathbb{Z})^\times} \varepsilon(d)\langle d\rangle\cdot 0. \tag{21}$$

Now suppose $N = ST$ with $(S,T)=1$. Let ε be a primitive Dirichlet character of conductor N and factor

$$\varepsilon = \varepsilon_S\varepsilon_T\colon (\mathbb{Z}/N\mathbb{Z})^\times \cong (\mathbb{Z}/S\mathbb{Z})^\times \times (\mathbb{Z}/T\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times.$$

For an elliptic curve E we identify the data (E, x_N) where x_N is a point of exact order N with $(E, x_S = Tx_N, x_T = Sx_N)$. Diamond operators $\langle d\rangle_S$ then act by

$$\langle d\rangle_S(E,x_S,x_T) = (E, dx_S, x_T).$$

By definition, $\langle d\rangle = \langle d\rangle_S\langle d\rangle_T$. It is easily checked that $w_{\zeta_S}\circ\langle d\rangle_S = \langle d^{-1}\rangle_S\circ w_{\zeta_S}$ and $w_{\zeta_S}\circ\langle d\rangle_T = \langle d\rangle_T\circ w_{\zeta_S}$. Factoring into S- and T- components we see from equation (20) that

$$\begin{aligned} a_0(E(\varepsilon,1)) &= (-1)^k i^k(2\pi)^{-k}\Gamma(k)L(k,\varepsilon)\textstyle\sum_{(a,b)}\varepsilon_S(a)\varepsilon_T(b)\langle a\rangle_S\langle b\rangle_T\cdot 0\\ &= \tfrac{(-1)^k\tau(\varepsilon)L(1-k,\varepsilon^{-1})}{2N^k}\textstyle\sum_{(a,b)}\varepsilon_S(a)\varepsilon_T(b)\langle a\rangle_S\langle b\rangle_T\cdot 0 \text{ if } \varepsilon \text{ is primitive.}\end{aligned} \tag{22}$$

Here the sum is taken over $(a,b)\in(\mathbb{Z}/S\mathbb{Z})^\times\times(\mathbb{Z}/T\mathbb{Z})^\times$. For any oriented cusp c we have from Proposition 3.5 that

$$a_0(w_{\zeta_S}E(\varepsilon,1);c) = (1/S)a_0(E(\varepsilon,1); w_{\zeta_S}c).$$

Hence $a_0(w_{\zeta_S}E(\varepsilon,1);c)=0$, unless $S = N_m(c)$ and $T = N_e(c)$, i.e., unless

$$c = \langle a\rangle_S\langle b\rangle_T\langle(\zeta_S,0);(1,1/T)\rangle \quad \text{for some } (a,b)\in(\mathbb{Z}/S\mathbb{Z})^\times\times(\mathbb{Z}/T\mathbb{Z})^\times.$$

But we compute

$$\begin{aligned} &a_0(w_{\zeta_S}E(\varepsilon,1);\langle a\rangle_S\langle b\rangle_T\langle(\zeta_S,0);(1,1/T)\rangle)\\ &\quad= (1/S)a_0(E(\varepsilon,1);\langle a^{-1}\rangle_S\langle b\rangle_T w_{\zeta_S}\langle(\zeta_S,0);(1,1/T)\rangle)\\ &\quad= S^{k-1}a_0(E(\varepsilon,1);\langle a^{-1}\rangle_S\langle b\rangle_T 0)\\ &\quad= S^{k-1}\varepsilon_S^{-1}(a)\varepsilon_T(b)\frac{(-1)^k\tau(\varepsilon)L(1-k,\varepsilon^{-1})}{2N^k}.\end{aligned}$$

Hence we obtain the formula:

PROPOSITION 3.11:

$$\begin{aligned} &a_0(w_{\zeta_S}E(\varepsilon,1)) \\ &= (-1)^k i^k (2\pi)^{-k}\Gamma(k)L(k,\varepsilon)\sum_{(a,b)}\varepsilon_S^{-1}(a)\varepsilon_T(b)\langle a\rangle_S\langle b\rangle_T\langle(\zeta_S,0);(1,1/T)\rangle \\ &= \frac{(-1)^k\tau(\varepsilon)L(1-k,\varepsilon^{-1})}{2NT^{k-1}}\sum_{(a,b)}\varepsilon_S^{-1}(a)\varepsilon_T(b)\langle a\rangle_S\langle b\rangle_T\langle(\zeta_S,0);(1,1/T)\rangle \end{aligned}$$

if ε is primitive.

To shorten notation set $f = w_{\zeta_S}E(\varepsilon,1)$. Then the Hecke eigenvalues of f are as follows.

PROPOSITION 3.12: *For $f = w_{\zeta_S}E(\varepsilon,1)$,*

(1) $\langle d\rangle f = \varepsilon_S^{-1}(d)\varepsilon_T(d)f$.
(2) $T_\ell f = (\varepsilon_T(\ell) + \varepsilon_S^{-1}(\ell)\ell^{k-1})f$ if $\ell \nmid N$.
(3) $U_\ell f = \varepsilon_S^{-1}(\ell)\ell^{k-1}f$ if $\ell | T$.
(4) *If ε_S is primitive, then* $U_\ell f = \varepsilon_T(\ell)f$ if $\ell | S$.

Proof: As usual, Proposition 3.5 reduces the problem to the action of the Hecke operators on the oriented cusps and this is in turn computed in Proposition 3.4. For (1), note that since $a_0(\langle d\rangle f;c) = a_0(f;\langle d\rangle c)$ for an oriented cusp c Proposition 3.11 shows that $a_0(\langle d\rangle f;c) = 0$ unless $c = \langle a\rangle_S\langle b\rangle_T\langle(\zeta_S,0);(1,1/T)\rangle$ for some $(a,b) \in (\mathbb{Z}/S\mathbb{Z})^\times \times (\mathbb{Z}/T\mathbb{Z})^\times$. But

$$\begin{aligned} &a_0(\langle d\rangle f;\langle a\rangle_S\langle b\rangle_T\langle(\zeta_S,0);(1,1/T)\rangle) \\ &\qquad = a_0(f;\langle da\rangle_S\langle db\rangle_T\langle(\zeta_S,0);(1,1/T)\rangle) \\ &\qquad = \varepsilon_S^{-1}(d)\varepsilon_T(d)a_0(f;\langle a\rangle_S\langle b\rangle_T\langle(\zeta_S,0);(1,1/T)\rangle). \end{aligned}$$

Hence $a_0(\langle d\rangle f) = a_0(\varepsilon_S^{-1}(d)\varepsilon_T(d)f)$, establishing (1).

For (2), we recall the formula (Proposition 3.5) $a_0(T_\ell f;c) = (1/\ell)a_0(f;T_\ell c)$ for any oriented cusp c. Appealing to the formula of Proposition 3.11 we deduce that $a_0(f;c) = 0$ unless $c = \langle a\rangle_S\langle b\rangle_T\langle(\zeta_S,0);(1,1/T)\rangle$. Moreover

$$\begin{aligned} &a_0(f;T_\ell\langle a\rangle_S\langle b\rangle_T\langle(\zeta_S,0);(1,1/T)\rangle) \\ &= (1/\ell)a_0(f;\langle a\rangle_S\langle b\rangle_T T_\ell\langle(\zeta_S,0);(1,1/T)\rangle) \\ &= \ell^{k-1}a_0(f;\langle(\zeta_S^{a\ell},0);(1,b/T)\rangle \\ &\quad + a_0(f;\langle(\zeta_S^{b},0);(1,c\ell/T)\rangle) \qquad \text{by Proposition 3.4} \\ &= (\ell^{k-1}\varepsilon_S^{-1}(\ell) \\ &\quad + \varepsilon_T(\ell))a_0(f;\langle a\rangle_S\langle b\rangle_T\langle(\zeta_S,0);(1,1/T)\rangle). \end{aligned}$$

We therefore have that

$$a_0(T_\ell f) = a_0((\ell^{k-1}\varepsilon_S^{-1}(\ell) + \varepsilon_T(\ell)f),$$

proving (2).

The proof of (3) in the case $\ell|T$ is exactly analogous to the proof of 2) above for the action of T_ℓ and is left to the reader. In case $\ell|S$, $a_0(f; U_\ell c) = 0$, unless $S_m(c) = S$ or $S_m(c) = S/\ell$, i.e., unless

$$c = \langle a\rangle_S\langle b\rangle_T\langle(\zeta_S, 0); (1, 1/T)\rangle \text{ or } c = \langle a\rangle_S\langle b\rangle_T\langle(\zeta_S, i/\ell); (1, 1/T)\rangle, \quad 1 \le i < \ell.$$

Now by Proposition 3.4,

$$U_\ell\langle(\zeta_S, 0); (1, 1/T)\rangle = \ell\langle(\zeta_S, 0); (1, \ell/T)\rangle.$$

Hence

$$\begin{aligned} a_0(U_\ell f; \langle a\rangle_S\langle b\rangle_T\langle(\zeta_S, 0); (1, 1/T)\rangle) =& a_0(f; \langle a\rangle_S\langle b\rangle_T U_\ell\langle(\zeta_S, 0); (1, 1/T)\rangle) \\ =& \ell\varepsilon_T(\ell) a_0(f; \langle a\rangle_S\langle b\rangle_T\langle(\zeta_S, 0); (1, 1/T)\rangle). \end{aligned}$$

On the other hand, for the oriented cusp $\langle(\zeta_S, i/\ell); (1, 1/T)\rangle$, $1 \le i < \ell$, we have from Proposition 3.4

$$U_\ell\langle(\zeta_S, i/\ell); (1, 1/T)\rangle = \sum_{\zeta_\ell^\ell = 1}\langle(\zeta_S\zeta_\ell^{-i}, 0); (1, \ell/T)\rangle \text{ if } (\ell, S/\ell) = \ell,$$

$$\begin{aligned} U_\ell\langle(\zeta_S, i/\ell); (1, 1/T)\rangle =& \ell^k\langle(\zeta_S^\ell, i/\ell); (1, \ell/T)\rangle \\ &+ \sum_{\substack{\zeta_\ell^\ell = 1 \\ \zeta_\ell^{iS_m} \ne \zeta_S^{S/\ell}}} \langle(\zeta_S\zeta_\ell^{-i}, 0); (1, \ell/T)\rangle \text{ if } (\ell, S/\ell) = 1. \end{aligned}$$

Hence we compute using the primitivity of ε_S

$$a_0(U_\ell f; \langle a\rangle_S\langle b\rangle_T\langle(\zeta_S, i/\ell); (1, 1/T)\rangle) = a_0(f; \langle a\rangle_S\langle b\rangle_T U_\ell\langle(\zeta_S, i/\ell), (1, 1/T)\rangle)$$

$$= \varepsilon_T(\ell) a_0(f; \langle a\rangle_S\langle b\rangle_T\langle(\zeta_S, 0); (1, 1/T)\rangle \sum_{\substack{x \in (\mathbb{Z}/S\mathbb{Z})^\times \\ x \equiv 1 \pmod{S/\ell}}} \varepsilon_S(x) = 0.$$

We have now shown that $a_0(U_\ell f) = a_0(\varepsilon_T(\ell)f)$ in case $\ell|S$ and ε is primitive. This then establishes (4) and the proof of the proposition is complete. ∎

We will need the q-expansion of f at ∞, and its computation involves the operator "Frobenius" operator V_ℓ on modular forms. Suppose $g \in M_k(\Gamma_1(N))$,

$\ell|N$, and $g(\infty)(q) = \sum a_n q^n$. Then V_ℓ acts by $V_\ell(g)(\infty)(q) = \sum a_n q^{n\ell}$. In the proposition below we give the moduli interpretation of V_ℓ which immediately implies that $V_\ell g \in M_k(\Gamma_1(N\ell))$. The formulae in terms of q-expansions show that $g - V_\ell U_\ell g$ is annihilated by U_ℓ for all $g \in M_k(\Gamma_1(N)$.

PROPOSITION 3.13: *Let ℓ be a prime. Define π_ℓ: $X_1(N\ell) \longrightarrow X_1(N)$ by $\pi_\ell(E,x) = (E/\langle Nx\rangle, \overline{x})$ for x a point of the elliptic curve E of exact order $N\ell$. Then for $g \in M_k(\Gamma_1(N))$:*

(1) *$V_\ell(g) = \ell^{-(k-1)}\pi_\ell^* g$.*

(2) *Suppose $\ell S|N$ with $(S,\ell) = 1$. Then*

$$a_0(V_\ell(g); \langle\zeta_{N\ell}, 1/S\ell\rangle) = a_0(g; \langle\zeta_N, 1/S\ell\rangle).$$

(3) *Suppose L is square-free and $SL|N$ with $(S,L) = 1$. Then*

$$a_0(\textstyle\prod_{\ell|L} V_\ell g; \langle\zeta_{NL}, 1/SL\rangle) = a_0(g; \langle\zeta_N, 1/SL\rangle).$$

Proof: For (1), we compute

$$\begin{aligned}
&\ell^{-(k-1)}(\pi_\ell^* g)(\infty)(q)\left(\tfrac{dq}{q}\right) \otimes \left(\tfrac{dt}{t}\right)^{\otimes(k-2)} \\
&= \ell^{-(k-1)} g(\mathbb{G}_m/\langle\zeta_\ell, q^{\mathbb{Z}}\rangle, \zeta_{N\ell}))(q)\left(\tfrac{dq}{q}\right) \otimes \left(\tfrac{dt}{t}\right)^{\otimes(k-2)} \\
&= \ell^{-(k-1)} g(\mathbb{G}_m/q^{\ell\mathbb{Z}}, \zeta_N)(q)\left(\tfrac{dq^\ell}{q^\ell}\right) \otimes \left(\tfrac{dt^\ell}{t^\ell}\right)^{\otimes(k-2)} \\
&= g(\infty)(q^\ell)\left(\tfrac{dq}{q}\right) \otimes \left(\tfrac{dt}{t}\right)^{\otimes(k-2)},
\end{aligned}$$

establishing that $V_\ell(g) = \ell^{-(k-1)}\pi_\ell^* g$.

For (2) we have

$$\begin{aligned}
&V_\ell(g)(\langle\zeta_{N\ell}, 1/S\ell\rangle)(q)\left(\tfrac{dq}{q}\right) \otimes \left(\tfrac{dt}{t}\right)^{\otimes(k-2)} \\
&= \ell^{-(k-1)}\pi_\ell^*(g)(\mathbb{G}_m/q^{\mathbb{Z}}, \zeta_{N\ell} q^{1/S\ell}) \\
&= \ell^{-(k-1)} g(\mathbb{G}_m/\langle q^{\mathbb{Z}}, \zeta_\ell\rangle, \zeta_{N\ell} q^{1/S\ell})\left(\tfrac{dq}{q}\right) \otimes \left(\tfrac{dt}{t}\right)^{\otimes(k-2)} \\
&= \ell^{-(k-1)} g(\mathbb{G}_m/q^{\ell\mathbb{Z}}, \zeta_N (q^\ell)^{1/S\ell})\left(\tfrac{dq^\ell}{q^\ell}\right) \otimes \left(\tfrac{dt^\ell}{t^\ell}\right)^{\otimes(k-2)} \\
&= g(\zeta_N, 1/S\ell)(q^\ell).
\end{aligned}$$

Hence $a_0(V_\ell g; (\zeta_{N\ell}, 1/S\ell)) = a_0(g; (\zeta_N, 1/S\ell))$. The assertion (3) then follows from (2) by induction. ∎

Recall now that $f = w_{\zeta_S} E(\varepsilon, 1)$. For ε primitive set

$$h = \prod_{\ell | S} (1 - \varepsilon_T(\ell) V_\ell) f.$$

Then h is a modular form of weight k on $\Gamma_1(N \prod_{\ell|S} \ell)$.

PROPOSITION 3.14:

$$\begin{aligned} \langle d \rangle h =& \varepsilon_S^{-1}(d) \varepsilon_T(d) h \quad \text{for } (d, N) = 1, \\ T_\ell h =& (\varepsilon_T(\ell) + \varepsilon_S^{-1}(\ell) \ell^{k-1}) h \quad \text{for } \ell \nmid N, \\ U_\ell h =& \begin{cases} \varepsilon_S^{-1}(\ell) \ell^{k-1} h & \text{for } \ell | T, \\ 0 \quad \text{for } \ell | S & \text{if } \varepsilon_S \text{ is primitive.} \end{cases} \end{aligned}$$

Assume ε_S is primitive and set $H = E(\varepsilon, 1) \otimes \varepsilon_S^{-1}$. The function H is a modular form of weight k on $\Gamma_1(NS)$. Computing its Hecke eigenvalues via Proposition 3.7 we see that they are the same as the eigenvalues of h given in Proposition 3.14, and hence

(23) $$h = \lambda H \quad \text{for some constant } \lambda.$$

Recall that the modular form h is of level $N \prod_{\ell|S} \ell$. In the above we view it as having the larger level NS via the *forgetful* map. The forgetful map is the natural covering $X_1(M') \longrightarrow X_1(M)$ defined for positive integers $M | M'$ in terms of moduli by $(E, x) \mapsto (E, (M'/M)x)$ where E is an elliptic curve with a point x of exact order M'.

To compute this constant λ, consider the oriented cusp $\langle \zeta_{NS}, 1/N \rangle$ of $X_1(NS)$ above (via the forgetful map) the oriented cusp

$$\langle \zeta_N, 1/T \rangle = \langle (\zeta_S, 0); (\zeta_T, S/T) \rangle = \langle (\zeta_S, 0); (1, S/T) \rangle$$

of $X_1(N)$. Since $a_0(f; c) = 0$ for any oriented cusp c of $X_1(N)$ for which $S_m(c) \neq S$, we have:

(24) $$a_0(h; \langle \zeta_{NS}, 1/N \rangle) = a_0(f; \langle \zeta_{NS}, 1/N \rangle) \quad \text{by Proposition 3.13, (3)}$$

(25) $$= a_0(f; \langle (\zeta_S, 0); (\zeta_T, S/T) \rangle$$

(26) $$= \frac{(-1)^k \tau(\varepsilon) L(1-k, \varepsilon^{-1})}{2NT^{k-1}} \varepsilon_T(S) \quad \text{by Proposition 3.11.}$$

We now determine $a_0(H; (\zeta_{NS}, 1/N))$. This involves tracing through the definition of twisting given in Definition 3.6; we employ the notation used there for easy comparison. Set $F = E(\varepsilon, 1)$; F is then a modular form of level N. The Dirichlet character ε_S^{-1} has conductor S. Let the pair (E, x) consist of an elliptic curve E together with an exact NS-division point x. The elliptic curve $E' = E/\langle Nx\rangle$ possesses exact S-division points $T\overline{x}$ and $\overline{y}$ with $\langle Nx, y\rangle = \zeta_S$. For $\lambda \in \mathbb{Z}/S\mathbb{Z}$, we have the isogeny $\varphi_\lambda\colon E \longrightarrow E_\lambda = E'/\langle\overline{y} + \lambda S\overline{x}\rangle$. Put $x_\lambda = \varphi_\lambda(x)$. Then

$$H(E,x) = 1/S^{k+1} \sum_{\lambda\in\mathbb{Z}/S\mathbb{Z}} \sum_{\mu\in(\mathbb{Z}/S\mathbb{Z})^\times} \varepsilon_S^{-1}(\mu)\zeta_S^{-\mu\lambda}\varphi_\lambda^* F(\varphi_\lambda(E,x)). \tag{27}$$

For $(E,x) = (\mathbb{G}_m/q^{\mathbb{Z}}, \zeta_{NS}q^{1/N})$, we have

$$\begin{aligned} E' &= E/\langle Nx\rangle = \mathbb{G}_m/\langle q^{\mathbb{Z}},\, \zeta_S\rangle \xrightarrow{(\ )^S} \mathbb{G}_m/q^{S\mathbb{Z}},\\ \overline{x} &= \zeta_N q^{1/T},\\ \overline{y} &= q \quad \text{since } \langle Nx, q^{1/S}\rangle = \zeta_S. \end{aligned} \tag{28}$$

We now proceed to find (E_λ, x_λ) associated to $(E,x) = (\mathbb{G}_m/q^{\mathbb{Z}}, \zeta_{SN}q^{1/N})$.

$$E_\lambda = E'/\langle\overline{y} + \lambda T\overline{x}\rangle = \mathbb{G}_m/\langle q^{S\mathbb{Z}},\, q(\zeta_N q^{1/T})^{T\lambda}\rangle = \mathbb{G}_m/\langle q^{S\mathbb{Z}}, \zeta_S^\lambda q^{\lambda+1}\rangle. \tag{29}$$

Correspondingly we have $x_\lambda = \zeta_N q^{1/T}$. Substituting into equation (27) we obtain

$$\begin{aligned} &H(\mathbb{G}_m/q^{\mathbb{Z}}, \zeta_{NS}q^{1/N}) = \\ &\quad 1/S \sum_{\lambda\in\mathbb{Z}/S\mathbb{Z}} \sum_{\mu\in(\mathbb{Z}/S\mathbb{Z})^\times} \varepsilon_S^{-1}(y)\zeta_S^{-\mu\lambda} F(\mathbb{G}_m/\langle q^{S\mathbb{Z}}, \zeta_S^\lambda q^{\lambda+1}\rangle, \zeta_N q^{1/T}). \end{aligned} \tag{30}$$

However, we know that $F = E(\varepsilon, 1)$ has zero constant term evaluated at the oriented cusp

$$\langle\mathbb{G}_m/\langle q^{S\mathbb{Z}}, \zeta_S^\lambda q^{\lambda+1}\rangle, \zeta_N q^{1/T})$$

unless $\zeta_N q^{1/T}$ is étale, i.e. its order modulo roots of unity is N. This in turn happens exactly when $\lambda + 1 \equiv 0 \bmod S$. Hence

$$\begin{aligned} a_0(H;\langle\zeta_{NS}, 1/N\rangle) &= 1/S \sum_{\mu\in(\mathbb{Z}/S\mathbb{Z})^\times} \varepsilon_S^{-1}(\mu)\zeta_S^\mu a_0(F;\langle\mathbb{G}_m/\langle q^{S\mathbb{Z}}, \zeta_S^{-1}\rangle, \zeta_N q^{1/T}\rangle) \\ &= S^{k-1} \sum_{\mu\in(\mathbb{Z}/S\mathbb{Z})^\times} \varepsilon_S^{-1}(\mu)\zeta_S^\mu a_0(F;\langle\mathbb{G}_m/q^{S^2\mathbb{Z}}, \zeta_T(q^{S^2})^{1/N}\rangle) \\ &= \frac{S^{k-1}\tau(\varepsilon_S^{-1})(-1)^k\tau(\varepsilon)L(1-k,\varepsilon^{-1})}{2N^k} \qquad \text{by equation (19)} \\ &= \frac{\varepsilon_S(T)\varepsilon_T(-S)\tau(\varepsilon_T)L(1-k,\varepsilon^{-1})}{2T^k}. \end{aligned} \tag{31}$$

For the last line above use the identities $\tau(\varepsilon_S)\tau(\varepsilon_S^{-1}) = S \cdot \varepsilon_S(-1)$ and

$$\tau(\varepsilon) = \tau(\varepsilon_S\varepsilon_T) = \tau(\varepsilon_S)\tau(\varepsilon_T)\varepsilon_T(S)\varepsilon_S(T).$$

We are now in a position to solve for the constant λ such that $h = \lambda H$ as in equation (23).

$$\begin{aligned} &h=\lambda H \\ &a_0(h;\langle\zeta_{NS}, 1/N\rangle)=\lambda a_0(H;\langle\zeta_{NS}, 1/N\rangle) \\ &\frac{(-1)^k\tau(\varepsilon)\varepsilon_T^{-1}(S)L(1-k,\varepsilon^{-1})}{2NT^{k-1}}=\lambda\frac{\varepsilon_S(T)\varepsilon_T(-S)\tau(\varepsilon_T)L(1-k,\varepsilon^{-1})}{2T^k}, \end{aligned} \tag{32}$$

the last line using equations (24) and (31). Solving for λ again using the identity $\tau(\varepsilon) = \tau(\varepsilon_S)\tau(\varepsilon_T)\varepsilon_T(S)\varepsilon_S(T)$, we obtain

$$\lambda = \lambda(\varepsilon, S) = \frac{(-1)^k\tau(\varepsilon_S)}{S\varepsilon_T(-S)}. \tag{33}$$

We now determine the q-expansion of $f = w_{\zeta_S}E(\varepsilon, 1)$ at ∞.

Proposition 3.15: *Set $L(s, \varepsilon_T)L(s+1-k, \varepsilon_S^{-1}) = \sum_{n=1}^{\infty} a_n n^{-s}$. Then*

$$w_{\zeta_S}E(\varepsilon,1)(\infty)(q) = \frac{(-1)^k\tau(\varepsilon_S)}{S\varepsilon_T(-S)}\left[-L(1-k,\varepsilon_S^{-1})L(0,\varepsilon_T) + \sum_{n=1}^{\infty} a_n q^n\right].$$

Proof: Using the notation of the above discussion,

$$a_1(w_{\zeta_S}E(\varepsilon,1);\infty) = a_1(h;\infty) = \lambda a_1(H;\infty) = \lambda.$$

Hence by Proposition 3.12

$$w_{\zeta_S}E(\varepsilon,1)(\infty)(q) = a_0(w_{\zeta_S}E(\varepsilon,1);\infty) + \lambda\sum_{n=1}^{\infty} a_n q^n.$$

The proposition then follows from the determination of $a_0(w_{\zeta_S}E(\varepsilon,1);\infty)$ given in Proposition 3.11. Recall that for a primitive Dirichlet character χ we have for positive integers k that $L(1-k,\chi) \neq 0$ if and only if $(-1)^k = \operatorname{sgn}(\chi)$ with the one exception $\zeta(0) = -1/2$ occurring for the trivial character. Note that this then gives $L(1-k, \varepsilon_S^{-1})L(0,\varepsilon_T) = 0$ unless $S = N$. ∎

Denote the conductor of a Dirichlet character χ by

$$\operatorname{cond}(\chi) = \prod_{\ell \text{ prime}} \ell^{\operatorname{cond}_\ell(\chi)}.$$

In general given a pair of Dirichlet characters (α, β) we now construct a normalized Eisenstein series $E(\alpha, \beta)$ of weight k and conductor $N_{\alpha,\beta} = \mathrm{cond}\,(\alpha)\mathrm{cond}\,(\beta)$ as follows. Define S' and T' with $S'T' = N_{\alpha,\beta}$, $(S', T') = 1$ by the requirement:

$$\begin{cases} \mathrm{cond}_\ell(\beta) > \mathrm{cond}_\ell(\alpha) & \text{for primes } \ell | S' \\ \mathrm{cond}_\ell(\beta) \le \mathrm{cond}_\ell(\alpha) & \text{for primes } \ell | T'. \end{cases} \tag{34}$$

Then we can factor the characters α and β:

$$\alpha = \alpha_{S'}\alpha_{T'}, \quad \beta = \beta_{S'}\beta_{T'}\colon (\mathbb{Z}/N_{\alpha,\beta}\mathbb{Z})^\times \cong (\mathbb{Z}/S'\mathbb{Z})^\times \times (\mathbb{Z}/T'\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times .$$

Defining $S = \mathrm{cond}\,(\beta_{S'})$, $T = \mathrm{cond}\,(\alpha_{T'})$ and adhering to the definition of λ in Equation (33), set

Definition 3.16: $E(\alpha, \beta) = \frac{1}{\lambda(\alpha\beta^{-1}, S)} w_{\zeta_S} E(\alpha\beta^{-1}, 1) \otimes \alpha_{S'}\beta_{T'}$.

We remark that $E(\alpha, \beta)$ is well-defined vis-a-vis our definitions of partial w-operators and twisting. Firstly note that $S | \mathrm{cond}\,(\alpha\beta^{-1})$ so that $w_{\zeta_S} E(\alpha\beta^{-1}, 1)$ is defined. Next $\mathrm{cond}\,(\alpha\beta^{-1}) | \mathrm{cond}\,(\alpha_{T'})\mathrm{cond}\,(\beta_{S'})$, so $E(\alpha\beta^{-1}, 1)$ is a modular form of level

$$N \stackrel{\text{def}}{=} \mathrm{cond}\,(\alpha_{T'}\beta_{S'}) = \mathrm{cond}\,(\alpha_{T'})\mathrm{cond}\,(\beta_{S'}) = ST.$$

Finally $M \stackrel{\text{def}}{=} \mathrm{cond}\,(\alpha_{S'}\beta_{T'}) = \mathrm{cond}\,(\alpha_{S'})\mathrm{cond}\,(\beta_{T'})$ divides N so the twisting

$$w_{\zeta_S} E(\alpha\beta^{-1}, 1) \otimes \alpha_{S'}\beta_{T'}$$

is a modular form of level

$$MN = \mathrm{cond}\,(\alpha_{S'})\mathrm{cond}\,(\alpha_{T'})\mathrm{cond}\,(\beta_{S'})\mathrm{cond}\,(\beta_{T'}) = \mathrm{cond}\,(\alpha)\mathrm{cond}\,(\beta) = N_{\alpha,\beta}.$$

Note that if $\mathrm{cond}_\ell(\alpha), \mathrm{cond}_\ell(\beta) > 0$ for a prime $\ell | N_{\alpha,\beta}$ then the Nebentypus character $\varepsilon = \alpha\beta$ of $E(\alpha, \beta)$ is not primitive. The essential properties of $E(\alpha, \beta)$ are given below.

THEOREM 3.17: *Set* $N_{\alpha,\beta} = \mathrm{cond}\,(\alpha)\mathrm{cond}\,(\beta)$.

(1) $\langle d\rangle E(\alpha, \beta) = \alpha\beta(d) E(\alpha, \beta)$,
$T_\ell(E(\alpha, \beta)) = (\alpha(\ell) + \beta(\ell)\ell^{k-1}) E(\alpha, \beta)$ for all $\ell \nmid N_{\alpha,\beta}$,
$U_\ell(E(\alpha, \beta)) = (\alpha(\ell) + \beta(\ell)\ell^{k-1}) E(\alpha, \beta)$ for all $\ell | N_{\alpha,\beta}$.

(2) *If* $L(s, \alpha)L(s + 1 - k, \beta) = \sum_{n=1}^\infty a_n n^{-s}$, *then*

$$E(\alpha, \beta)(\infty)(q) = L(0, \alpha)L(1 - k, \beta) + \sum_{n=1}^\infty a_n q^n.$$

Proof: (1) follows from Proposition 3.12 and Proposition 3.7.

As for (2), applying Proposition 3.15 and equation (14) to the definition of $E(\alpha, \beta)$ (3.16) yields

$$E(\alpha,\beta)(\infty)(q) = -L(1-k, \alpha_S^{-1}\beta_S)L(0, \alpha_T\beta_T^{-1})(\alpha_S\beta_T)(0) + \sum_{n=1}^{\infty} a_n q^n,$$

where $L(s,\alpha)L(s+1-k,\beta) = \sum_{n=1}^{\infty} a_n n^{-s}$. However observe that for characters χ_1, χ_2 and $k \geq 1$ with $(\chi_1\chi_2)(-1) = (-1)^k$ we have $L(0,\chi_1)L(1-k,\chi_2) = 0$ unless one of χ_1, χ_2 is trivial. From this it follows easily that

$$-L(1-k, \alpha_S^{-1}\beta_S)L(0, \alpha_T\beta_T^{-1})(\alpha_S\beta_T)(0) = L(0,\alpha)L(1-k,\beta),$$

concluding the proof. ∎

COROLLARY 3.18: *Suppose α and β are Dirichlet characters. Define*

$$E(\alpha,\beta)(q) = -L(0,\alpha)L(1-k,\beta) + \sum_{n=1}^{\infty} a_n q^n$$

where $q = e^{2\pi i z}$ and $L(s,\alpha)L(s+1-k,\beta) = \sum_{n=1}^{\infty} a_n n^{-s}$. Then $E(\alpha,\beta)$ is a modular form of weight k and level cond (α)cond (β).

Remark 3.19: A more conventional proof of Corollary 3.18 may be given by using the functional equations for $L(s,\alpha)$ and $L(s,\beta)$ to show that $E(\alpha,\beta)$ and its twists satisfy the correct functional equations. Hence $E(\alpha,\beta)$ defines a modular form by Weil's Converse Theorem. This is the approach taken in [19]. Constructing Eisenstein series via their constant terms as we have done here has the obvious advantage that it is possible to conceptually treat the constant terms at all of the cusps; this will be necessary for our subsequent consideration of Eisenstein ideals.

We lastly analyze the constant term of the Eisenstein series $E(\alpha,\beta)$.

THEOREM 3.20: *Let $\mathbb{Z}[\alpha,\beta]$ denote the finite extension of $\mathbb{Z}$ generated by the values of the characters α and β and set $R = \mathbb{Z}[1/(2N_{\alpha,\beta}), \alpha, \beta]$ with $N_{\alpha,\beta} =$* cond (α)cond (β).

(1) *The constant term $a_0(E(\alpha,\beta); c) = 0$ for any oriented cusp c with $N_e(c) < T$.*
(2) *Assume $(M,S) = 1$. Then the constant term $a_0(E(\alpha,\beta); c) = 0$ for any oriented cusp c with $S_e(c) \neq 1$.*

(3) *Assume* $(M,S)=1$. *Then the constant term*

$$a_0(E(\alpha,\beta);\langle\zeta_{NM},1/T\rangle)=uL(1-k,\alpha^{-1}\beta)$$

for a unit $u\in R^\times$.

(4) $a_0(E(\alpha,\beta))=L(1-k,\alpha^{-1}\beta)\cdot v$ *for a primitive vector* v *in the lattice* R[*oriented cusps*].

Proof: Let F be a modular form of level N and τ a Dirichlet character of conductor $M|N$. Suppose c is an oriented cusp of $X_1(MN)$ associated to the data (E,x) where E is a Tate curve and x is a point of E of exact order MN. Recall the definition of twisting (Definition 3.6). This constructs an isogeny $\phi_\lambda\colon(E,x)\longrightarrow(E_\lambda,x_\lambda)$ for each $\lambda\in(\mathbb{Z}/M\mathbb{Z})$. Let c_λ be the oriented cusp of $X_1(N)$ associated to (E_λ,x_λ). We have

$$a_0(F\otimes\tau;c)=\frac{1}{M^{k+1}}\sum_{\lambda\in\mathbb{Z}/M\mathbb{Z}}\sum_{\mu\in(\mathbb{Z}/M\mathbb{Z})^\times}\tau(\mu)\zeta_0^{-\mu\lambda}a_0(F;(\varphi_\lambda)_*c). \tag{35}$$

Consider $F=(1/\lambda(\alpha\beta^{-1},S))w_{\zeta_S}E(\alpha\beta^{-1},1)$ and $\tau=\alpha_{S'}\beta_{T'}$. Then $F\otimes\tau=E(\alpha,\beta)$ and by Proposition 3.11

$$a_0(F)=L(1-k,\alpha^{-1}\beta)\sum_{\text{oriented cusps } c}u_c c, \tag{36}$$

where each u_c is either 0 or a unit in R. Moreover $u_c=0$ unless $N_m(c)=S$ and $N_e(c)=T$ in the notation of the discussion preceding Theorem 3.17. Firstly from the definition of (E_λ,x_λ) we see that if $N_e(c)<T$ then $N_e(c_\lambda)<T$ for all $\lambda\in(\mathbb{Z}/M\mathbb{Z})$. This implies that $a_0(F;c_\lambda)=0$ for all $\lambda\in(\mathbb{Z}/M\mathbb{Z})$ and so $a_0(F;c)=0$, proving (1).

We now establish (2) and (3). So suppose we are in the case $(M,S')=1$, i.e., $\alpha_{S'}=1$ and $\tau=\beta_T$. Then since all the isogenies λ occurring in equation (35) are of degree prime to S we have $S_e(c)=S_e(c_\lambda)$, and hence (2) follows from equations (35) and (36). As for (3), let $c=\langle\zeta_{MN},1/T\rangle$. Then c is the oriented cusp associated to the data $(E,x)=(\mathbb{G}_m/q^{\mathbb{Z}},\zeta_{MN}q^{1/T})$. The computation is similar to that of equations (27)–(31). The main points are that $E'=\mathbb{G}_m/\langle q^{\mathbb{Z}},\zeta_M\rangle$ with the Mth power map identifying this with $\mathbb{G}_m/q^{M\mathbb{Z}}$. Accordingly then $\overline{x}=\zeta_N q^{M/T}$ and $\overline{y}=q$. We have $E_\lambda=\mathbb{G}_m/\langle q^{M\mathbb{Z}},\zeta_M^\lambda q^{1+\lambda s}\rangle$ with $x_\lambda=\zeta_N q^{M/T}$. The group generated by x_λ has order modulo roots of unity equal to T if and only if M/T has order T modulo $\langle M,1+\lambda s\rangle$. This in turn happens exactly when $\lambda \cong -1/S \bmod M$. Hence $a_0(F; c_\lambda) = 0$ for $\lambda \not\cong (-1/S) \bmod M$ and $a_0(F; c_{(-1/S)}) = u'L(1-k, \alpha^{-1}\beta)$ for $u' \in R^\times$ by equation (36). Hence from equation (35) we have

$$(37)\ a_0(F; c) = uL(1-k, \alpha^{-1}\beta) \quad \text{where } u = 1/M\Big(\sum_{\mu \in (\mathbb{Z}/M\mathbb{Z})^\times} \tau(\mu)\zeta_0^{-\mu/S}\Big)u'.$$

The term in parentheses is just a Gauss sum associated to τ and so $u \in R^\times$, proving (3).

For (4), note that from equations (35) and (36) it follows that

$$a_0(E(\alpha,\beta)) = L(1-k, \alpha^{-1}\beta) \cdot v \quad \text{for some vector } v \in R[\textit{oriented cusps}].$$

By (3), this vector v is known to be primitive in case $(M, S) = 1$. We will reduce the general case to this special case using the formula

$$(38) \qquad E(\alpha,\beta) \otimes \alpha_S^{-1} = \Big(\prod_{\ell|(M,S')} (1 - V_\ell U_\ell)\Big) F \otimes \beta_{T'}.$$

To see this look at q-expansions at ∞ using equation (14). If $F(\infty)(q) = \sum_{n=0}^\infty a_n q^n$, then

$$[E(\alpha,\beta) \otimes \alpha_{S'}^{-1}](\infty)(q) = [(F \otimes \tau) \otimes \alpha_{S'}^{-1}](q)$$

$$= \sum_{n=0}^{\infty} a_n \alpha_{S'}(n)\beta_{T'}(n)\alpha_{S'}^{-1}(n)q^n = \sum_{\substack{n=0 \\ (n,(M,S'))=1}} a_n \beta_{T'}(n) q^n.$$

But $(F \otimes \beta_{T'})(\infty)(q) = \sum_{n=0}^\infty a_n \beta_{T'}(n) q^n$, yielding (38).

Now note the formulae

$$U_\ell(F \otimes \beta_{T'}) = (\alpha_T(\ell) + (\alpha_S^{-1}\beta)(\ell)\ell^{k-1})F \otimes \beta_{T'} \quad \text{by Theorem 3.17}$$

$$\text{and} \quad V_\ell(F \otimes \beta_{T'}) = \ell^{-(k-1)}\pi_\ell^*(F \otimes \beta_{T'}) \quad \text{by Proposition 3.13, (1).}$$

Set $\gamma = (\alpha_S^{-1}\beta)(\ell) + \alpha_T(\ell)\ell^{-(k-1)}$. Then we have for an oriented cusp c

$$(39) \qquad a_0((1 - V_\ell U_\ell)F \otimes \beta_{T'}; c) = a_0(F \otimes \beta_{T'}, (\pi_\ell)_* c).$$

Setting

$$c' = \langle \zeta_{MN}, 1/T \prod_{\ell|(M,S')} 1/\ell \rangle,$$

we claim that

$$a_0((\prod_{\ell|(M,S')}(1-V_\ell U_\ell))F\otimes\beta_{T'};c') = a_0(F\otimes\beta_{T'};c'). \tag{40}$$

By equation (39), this will follow from the assertion that

$$a_0(F\otimes\beta_{T'},\langle\zeta_{NT},1/TL\rangle) = 0 \quad \text{for } 1\neq L|(M,S'). \tag{41}$$

But this in turn follows directly from (2).

We deduce that

$$a_0((\prod_{\ell|(M,S')}(1-V_\ell U_\ell))F\otimes\beta_{T'};c') = uL(1-k,\alpha^{-1}\beta)$$

for a unit u from equation (40) and (3) in Theorem 3.20. Hence

$$a_0(E(\alpha,\beta)\otimes\alpha_S^{-1}) = L(1-k,\alpha^{-1}\beta)\cdot v \quad \text{for a primitive vector } v.$$

This in turn means that $a_0(E(\alpha,\beta)) = L(1-k,\alpha^{-1}\beta)\cdot v$ for a primitive vector v, concluding the proof. ∎

3.2 Reducible Galois representations arising from cusp forms.

Fix a positive integer N and let $\mathbb{T}$ be the weight k Hecke algebra for $\Gamma_1(N)$ as defined in Section 2. Let $\mathbb{T}_0 = \mathbb{Z}\langle T_\ell, \ell\nmid N; \langle d\rangle, d\in(\mathbb{Z}/N\mathbb{Z})^\times\rangle\subseteq\mathbb{T}$ be the weight k *restricted* Hecke algebra. The full Hecke algebra $\mathbb{T}$ is then generated by $\mathbb{T}_0$ and U_ℓ, $\ell|N$. Suppose $\mathbf{m}\subseteq\mathbb{T}$ is a maximal ideal such that the corresponding Galois representation $\rho_{\mathbf{m}}$ is reducible. We will call such maximal ideals $\mathbf{m}\subseteq\mathbb{T}$ *reducible.*

Eisenstein series furnish examples of reducible maximal ideals. Let $R\subseteq\mathbb{C}$ be a finite extension of $\mathbb{Z}[\zeta_N]$ big enough to contain the values of all Dirichlet characters of conductor dividing N and set $\mathbb{T}'=\mathbb{T}\otimes R$, $\mathbb{T}_0'=\mathbb{T}_0\otimes R$. Let K be the quotient field of R. Suppose E is a normalized weight k Eisenstein series for $\Gamma_1(N)$ with character ε which is an eigenfunction of $\mathbb{T}$. Then

$$E(\infty)(q) = \sum_{n=0}^{\infty} a_n(E)q^n$$

with $T_\ell(E) = a_\ell(E)E$ for a prime $\ell\nmid N$ and $U_\ell(E)=a_\ell(E)E$ for a prime $\ell|N$. It is known that $a_\ell(E)\in R$ for all primes ℓ and $a_0(E)\in K[$*oriented cusps*$]$. Suppose

$\wp \subseteq R$ is a prime ideal with associated valuation $v_\wp$ such that $v_\wp(a_0(E;c)) > 0$ for each oriented cusp c of $X_1(N)$.

Definition 3.21: Set

$$\begin{aligned}
\mathbf{m}'_0(E,\wp) &= (T_\ell - a_\ell(E),\ \ell \nmid N;\ \langle d\rangle - \varepsilon(d),\ (d,N)=1;\ \wp) \subseteq \mathbb{T}'_0,\\
\mathbf{m}'(E,\wp) &= (\mathbf{m}_0(E,p),\ U_\ell - a_\ell(E),\ \ell | N) \subseteq \mathbb{T}',\\
\mathbf{m}_0(E,\wp) &= \mathbf{m}'_0 \cap \mathbb{T}_0,\\
\mathbf{m}(E,\wp) &= \mathbf{m}' \cap \mathbb{T}.
\end{aligned}$$

Note that the ideals defined above are indeed proper maximal ideals since $E \bmod \wp$ is a cusp form. The *Eisenstein* maximal ideals of $\mathbb{T}'$ (respectively $\mathbb{T}'_0$) are the ideals $\mathbf{m}'(E,\wp)$ (respectively $\mathbf{m}'_0(E,\wp)$), with E an Eisenstein series in $M_k(\Gamma_1(N))$ which is a $\mathbb{T}$-eigenform and $\wp \subseteq R$ a prime ideal dividing $a_0(E;c)$ for each oriented cusp c of $X_1(N)$. The *Eisenstein* maximal ideals of $\mathbb{T}$ (respectively $\mathbb{T}_0$) are those obtained by contracting Eisenstein maximal ideals of $\mathbb{T}'$ (respectively $\mathbb{T}'_0$). Note that all Hecke algebras are finite over $\mathbb{Z}$ (as modules), hence finite over each other whenever there is an inclusion. Hence all maximal ideals contract under the extensions of Hecke algebras considered to maximal ideals. Also all $\mathbf{m}' \subseteq \mathbb{T}$ containing a fixed $\mathbf{m} \subseteq \mathbb{T}$ are conjugate under Gal(R).

We can now state precisely the conjecture that all reducible representations arising from cusp forms should be detected from congruences with Eisenstein series.

CONJECTURE 3.22:

1. *A maximal ideal* $\mathbf{m}'_0 \subseteq \mathbb{T}'_0$ *is reducible if and only if it is Eisenstein.*
2. *A maximal ideal* $\mathbf{m}' \subseteq \mathbb{T}'$ *is reducible if and only if it is Eisenstein.*

Our analysis of reducible maximal ideals $\mathbf{m}'_0 \subseteq \mathbb{T}'_0$ (respectively $\mathbf{m}' \subseteq \mathbb{T}'$) will be facilitated by the notion of maximal ideals $\mathbf{m}_0 \subseteq \mathbb{T}_0$ (respectively $\mathbf{m} \subseteq \mathbb{T}$) which are *new* of level N.

Definition 3.23: Say that $\mathbf{m}'_0 \subseteq \mathbb{T}'_0$ is *new* in case the semi-simple representation $\rho_{\mathbf{m}'}$ of Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$ does not occur in cusp forms of any strictly lower level. Equivalently the eigenvalues for $\{T_\ell, \ell \nmid N;\ \langle d\rangle\}$ associated to $\mathbf{m}'$ occur for no maximal ideal of level $N'|N$, $N' \neq N$. Say that $\mathbf{m} \subseteq \mathbb{T}$ is *new* if the eigenvalues for $\{T_\ell, \ell \nmid N;\ \langle d\rangle;\ U_\ell, \ell|N'\}$ occur for no maximal ideal of level $N'|N$, $N' \neq N$. Lastly a maximal ideal $\mathbf{m} \subseteq \mathbb{T}$ (respectively $\mathbf{m}_0 \subseteq \mathbb{T}_0$) is *new* in case an extension $\mathbf{m}'$ of $\mathbf{m}$ (respectively $\mathbf{m}'_0$ of $\mathbb{T}'_0$) is new.

Suppose α and β are Dirichlet characters with $\operatorname{cond}(\alpha)\operatorname{cond}(\beta) = N$ and $\varepsilon = \alpha\beta$ satisfying $\varepsilon(-1) = (-1)^k$ for an integer $k \geq 2$. We then have $\alpha, \beta\colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow R^\times \subset \mathbb{C}^\times$. In Section 3.1 we constructed and studied the weight k Eisenstein series $E(\alpha,\beta)$ on $\Gamma_1(N')$ with character ε. Recall that $E(\alpha,\beta)$ is normalized and an eigenfunction for the Hecke operators $\{T_\ell, \ell \not| N;\ U_\ell, \ell|N;\ \langle d\rangle\}$; the eigenvalues are given in Theorem 3.17. Such an Eisenstein series will be called a *new* Eisenstein series of level N. The space of Eisenstein series of weight k for $\Gamma_1(N)$ is spanned by the new Eisenstein series $E(\alpha,\beta)$ of all levels $N'|N$ together with their push-ups under the natural degeneracy maps to level N. In particular if $\mathbf{m}_0$ is an Eisenstein ideal of $\mathbb{T}_0$ then $\mathbf{m}_0 = \mathbf{m}_0(E(\alpha,\beta),\wp)$ with $\operatorname{cond}(\alpha)\operatorname{cond}(\beta) = N'|N$. Let $\pi_\wp\colon R^\times \longrightarrow (R/\wp)^\times$ be the natural map and put $\overline{\alpha} = \pi_\wp \circ \alpha$, $\overline{\beta} = \pi_\wp \circ \beta$. We then have $\rho_{\mathbf{m}_0} = \overline{\alpha} \oplus \overline{\beta}\chi^{k-1}$ where χ is the p-cyclotomic character. Hence one direction of Conjecture 3.22 is clear – Eisenstein maximal ideals are always reducible. To classify reducible ideals we adopt the following notation:

Definition 3.24: Suppose α and β are Dirichlet characters with $\operatorname{cond}(\alpha)\operatorname{cond}(\beta)|N$. For an integer M define the ideals

$$\begin{aligned}
I(\alpha,\beta)'(M) =& \langle T_\ell - (\alpha(\ell) + \ell^{k-1}\beta(\ell)) \quad \text{for } \ell \not| MN;\\
& U_\ell - (\alpha(\ell) + \ell^{k-1}\beta(\ell)) \quad \text{for } \ell|N,\ \ell \not| M;\ \langle d\rangle - \varepsilon(d)\rangle \subseteq \mathbb{T}',\\
I(\alpha,\beta)_0'(M) =& \langle T_\ell - (\alpha(\ell) - \ell^{k-1}\beta(\ell)) \quad \text{for } \ell \not| MN;\ \langle d\rangle - \varepsilon(d)\rangle \subseteq \mathbb{T}_0',\\
I(\alpha,\beta)(M) =& I(\alpha,\beta)'(M) \cap \mathbb{T},\\
I(\alpha,\beta)_0(M) =& I(\alpha,\beta)_0'(M) \cap \mathbb{T}_0.
\end{aligned}$$

Furthermore we set $I(\alpha,\beta)' = I(\alpha,\beta)(1)$, $I(\alpha,\beta)_0' = I(\alpha,\beta)_0'(1)$, $I(\alpha,\beta) = I(\alpha,\beta)(1)$, and lastly $I(\alpha,\beta)_0 = I(\alpha,\beta)_0(1)$.

We can immediately say the following about reducible ideals.

PROPOSITION 3.25: *Let $\mathbb{T}$ be the weight k Hecke algebra for $\Gamma_1(N)$ and let $\mathbf{m} \subseteq \mathbb{T}$ be a reducible ideal of residue characteristic p with $\mathbf{m}_0 = \mathbf{m} \cap \mathbb{T}_0$. Suppose $\rho_{\mathbf{m}} = \overline{\alpha} \oplus \overline{\beta}\chi^{k-1}$, where χ is the cyclotomic character at p. Let $\wp$ be a prime of R of residue characteristic p and α, β: $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow R^\times \subset \mathbb{C}^\times$ Dirichlet characters such that $\overline{\alpha} = \pi_\wp \circ \alpha$, $\overline{\beta} = \pi_\wp \circ \beta$.*

(1) *$I(\alpha,\beta)(p)_0 \subseteq \mathbf{m}_0$.*

(2) *Suppose $p > k$ and $p \not| N$. Then $p \not| \operatorname{cond}(\alpha)\operatorname{cond}(\beta)$ and $I(\alpha,\beta)_0 \subseteq \mathbf{m}_0$.*

Proof: The representation $\rho_{\mathbf{m}}$ occurs in $H^1_{\mathrm{par}}(Y_1(N)_{\overline{\mathbb{Q}}}, \overline{\vartheta}_p)[\mathbf{m}]^\vee$. Hence (1) follows from the Eichler–Shimura relations. For (2) the crystalline theory is required. Specifically recall the definition $T_p^{\text{crys}} = F_p + \langle p \rangle F_p^t$, where F_p denotes Frobenius and F_p^t is its adjoint with respect to the inner product. Hence from the representation $\rho_{\mathbf{m}}$ we see that $T_p^{\text{crys}} = \alpha(p) + \beta(p)p^{k-1}$. But then T_p^{crys} is associated to $T_p^{\text{ét}}$ as in Theorem 1.2, (2). Hence $T_p - (\alpha(p) + \beta(p)p^{k-1}) \in \mathbf{m}\mathbb{T}'$, concluding the proof of the proposition. ∎

Hence for a reducible ideal $\mathbf{m}$ in the weight k Hecke algebra $\mathbb{T}$ for $\Gamma_1(N)$ what is unknown are the residue characteristic and the eigenvalues of the $U_\ell \bmod \mathbf{m}$. If $N = \ell^e N'$ with $(N', \ell) = 1$, then relating the eigenvalue of U_ℓ to Eisenstein series is a study of bad reduction. We are able to execute this program in two important cases. The first is the case when $e = 1$ and the Nebentypus character is trivial. This is the case of semi-stable reduction at ℓ where Picard–Lefschetz theory can be invoked. The second is the case of a Nebentypus character which is very ramified at ℓ. Here the Good Reduction Theorem of Katz–Mazur [11] can be applied. In the next two subsections we take up the analysis of these cases of bad reduction.

Note that to classify reducible modular Galois representations associated to cusp forms it suffices to classify those which are *new*. When analyzing the eigenvalues of U_ℓ associated to a reducible ideal $\mathbf{m} \subseteq \mathbb{T}$ and a prime of bad reduction ℓ we will systematically work under the simplifying assumption that $\mathbf{m}_0 \subseteq \mathbb{T}_0$ is new. If $\mathbf{m}$ is new at N, it is an open problem to classify all extensions to levels N' with $N|N'$.

3.3 Geometry mod ℓ: The case $e > 1$. Suppose $N = \ell^e \cdot N'$ with $(N', \ell) = 1$. For our purposes the best integral model of $Y_1(N)$ for studying the bad reduction arises from $Y_1^{\text{bal}}(N)_{/\mathbb{Z}[\zeta_N]}$, the moduli scheme for *balanced* $\Gamma_1(N)$-structures. A balanced $\Gamma_1(N)$-structure on an elliptic curve E/S with S any scheme is a diagram

$$P; E \underset{\pi^t}{\overset{\pi}{\rightleftarrows}} E'; Q,$$

where E' is an elliptic curve over S, π is an N-isogeny with dual isogeny π^t, and P, Q are Drinfeld bases of $\operatorname{Ker}\pi$, $\operatorname{Ker}\pi^t$ respectively. The curve $X_1^{\text{bal}}(N)$ is geometrically reducible but connected over $\mathbb{Z}$. The diamond operators $\langle\ \rangle \times \langle\ \rangle$ which we shall shortly define then operate semi-linearly on $X_1^{\text{bal}}(N)_{/\mathbb{Z}[\zeta_N]}$ since they exchange the ζ_N's. The curve $X_1^{\text{bal}}(N)_{/\mathbb{Z}[\zeta_N]}$ is reducible; any component is then a model of $X_1(N)$ over $\mathbb{Z}[\zeta_N]$.

We recall results of Katz–Mazur [11] concerning $Y_1^{\mathrm{bal}}(N)_{/\mathbb{Z}[\zeta_N]}$ and its special fiber X_s in characteristic ℓ. As notation, let F_ℓ denote a Frobenius element at ℓ and for a natural number m write $E^{(m)}$ for the image of the elliptic curve E under F_ℓ^m. For $g+h=e$, denote by $C_{g,h}$ the Igusa curve classifying elliptic curves E endowed with $\Gamma_1(N')$-level structure together with a generator W of $\mathrm{Ker}(V_\ell^{\max(g,h)}) \subseteq E^{\max(g,h)}$. The special fiber

$$X_s = \bigcup_{g+h=e} \left(\mathbb{Z}_\ell/\ell^{\min(g,h)}\mathbb{Z}_\ell\right)^\times \times C_{g,h}$$

with all the irreducible components $C_{g,h}$ meeting at the supersingular points. The geometric special fiber $X_{\overline{s}}$ is then the union of geometric Igusa curves $\overline{C}_{g,h} = C_{g,h} \times \overline{\mathbb{F}}_\ell$. Let V_ℓ denote Verschiebung at ℓ. The map

(42) $\qquad C_{g,h} \times \left(\mathbb{Z}_\ell/\ell^{\min(g,h)}\mathbb{Z}_\ell\right)^\times \longrightarrow X_s$ is defined as follows:

If $g \geq h$, $(E,W,u) \mapsto$ isogenies with Drinfeld generators P,Q:

$$E^{(g)} \xrightarrow{V_\ell^g} E \xrightarrow{F_\ell^h} E^{(h)} \quad P = W,\ Q = u \cdot V_\ell^{g-h}(W) \text{ and dual}$$
$$E^{(g)} \xleftarrow{F_\ell^g} E \xleftarrow{V_\ell^h} E^{(h)} \quad P = V_\ell^{g-h}W,\ Q = u \cdot W.$$

There is an action of $(\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ on $X_1^{\mathrm{bal}}(N)$ by letting $\langle a,b\rangle$ send the data $P; E \underset{\pi^t}{\overset{\pi}{\rightleftarrows}} E'; Q$ to $aP; E \underset{\pi^t}{\overset{\pi}{\rightleftarrows}} E'; bQ$. The action of $\langle a, a^{-1}\rangle$ is simply the diamond action $\langle a\rangle$ on $X_1(N)$. And the action of $\langle 1,b\rangle$ is via $(\mathbb{Z}/N\mathbb{Z})^\times = \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$. This is because the usual model of $Y_1(N)/\mathbb{Q}$ is obtained by dividing $Y_1^{\mathrm{bal}}(N)_{/\mathbb{Q}(\mu_N)}$ by the action of $\langle 1\,,\,b\rangle$. Then

$$Y_1(N)_{/\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = Y_1^{\mathrm{bal}}(N)_{/\mathbb{Q}(\mu_N)} \otimes_{\mathbb{Q}(\mu_N)} \overline{\mathbb{Q}}.$$

If $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the second factor of the first tensor product, it corresponds to $\langle 1\,,\,b\rangle \otimes_{\mathbb{Q}(\mu_N)} \sigma$ on the second factor (if $\sigma|_{\mathbb{Q}(\mu_N)} = b$). Hence on fibers over $\overline{\mathbb{F}}_\ell$, inertia at ℓ acts via $\langle 1\,,\,b\rangle$. Generally the action of $\langle a,b\rangle$ corresponds to $P \mapsto aP$, $u \mapsto ba^{-1}u$ if $g \geq h$ and $P \mapsto bP$, $u \mapsto ba^{-1}u$ if $h \geq g$. In particular the (g,h,u)-component of X_s is stabilized by the subgroup $K(g,h) = \{\langle a,b\rangle \,|\, ab^{-1} \equiv 1 \bmod \ell^{\min(g,h)}\}$. Let χ_1, χ_2: $(\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow (\mathbb{Z}/N\mathbb{Z})^\times$ denote projection onto the first and second factors, respectively. Set

$$\text{(43)}\quad \mathrm{Im}\,(K(g,h)) = \left\{ \begin{array}{ll} \chi_1(K(g,h)) \bmod \ell^g & \text{if } g \geq h \\ \chi_2(K(g,h)) \bmod \ell^h & \text{if } g \leq h. \end{array} \right\} = (\mathbb{Z}_\ell/\ell^{\max(g,h)}\mathbb{Z}_\ell)^\times.$$

Then $K(g,h)$ acts on a component $C_{g,h}$ of X_s through $\operatorname{Im} K(g,h)$.

Let ϕ: $E \longrightarrow Y_1(N)$ be the universal elliptic curve with compactification $\overline{\phi}$: $\overline{E} \longrightarrow X_1(N)$. Fix a prime $\not| N$ and recall the notation (for $k \geq 0$) $\vartheta_p = \vartheta_p(k) = \operatorname{Symm}^{k-2} R^1\overline{\phi}_*\mathbb{Z}_p$. The restrictions of the sheaf $\vartheta_p(k)$ to the various components of X_s will still be denoted by $\vartheta_p(k)$. Denote by $X_{\overline{s}}^{\mathrm{ord}}$ the ordinary locus of the geometric special fiber $X_{\overline{s}}$. Note that

$$H_!^1(X_{\overline{s}}^{\mathrm{ord}}, \vartheta_p(k)) \longrightarrow H^1(X_{\overline{s}}, \vartheta_p(k)) \text{ is surjective} \qquad \text{and}$$
$$H_!^1(X_{\overline{s}}^{\mathrm{ord}}, \vartheta_p(k)) = \prod_{g+h=e} H_!^1(\overline{C}_{g,h}^{\mathrm{ord}}, \vartheta_p(k)) \otimes \operatorname{Map}\left(\left(\mathbb{Z}_\ell/\ell^{\min(g,h)}\mathbb{Z}_\ell\right)^\times, \mathbb{Z}_\ell\right).$$
(44)

By Vanishing Cycle Theory there is an injection

$$H^1(X_{\overline{s}}, \vartheta_p(k)) \hookrightarrow H^1(X_{\overline{\eta}}, \vartheta_p(k)). \tag{45}$$

The Good Reduction Theorem of Katz–Mazur [11, Theorem 14.5.1] (adding auxiliary level) asserts that (45) is surjective on the ε-eigenspaces for the action of the diamond operators if $\operatorname{cond}_\ell(\varepsilon) > e/2$.

Now suppose $0 \neq z \in H_!^1(\overline{C}_{g,h}^{\mathrm{ord}}, \vartheta_p(k))$ is a cohomology class which transforms under $\langle a, a^{-1}\rangle$ via ε and under $\langle 1, b\rangle$ via $\alpha^{-1}(b)$. Then generally z transforms under $\langle a, b\rangle$ via $\varepsilon(a)/\alpha(ab)$. From the preceding discussion we see that

If $g \geq h$, then

| $\langle 1, b\rangle$ operates trivially if $b \equiv 1 \bmod \ell^h$ implying $\alpha \equiv 1$ on $1 + \ell^h\mathbb{Z}_\ell$.
| $\langle a, a^{-1}\rangle$ operates trivially if $a \equiv 1 \bmod \ell^g$ implying $\varepsilon \equiv 1$ on $1 + \ell^g\mathbb{Z}_\ell$.

If $h \geq g$, then

| $\langle a, 1\rangle$ operates trivially if $a \equiv 1 \bmod \ell^g$ implying $\varepsilon \equiv \alpha$ on $1 + \ell^g\mathbb{Z}_\ell$.
| $\langle a, a^{-1}\rangle$ operates trivially if $a \equiv 1 \bmod \ell^h$ implying $\varepsilon \equiv 1$ on $1 + \ell^h\mathbb{Z}_\ell$.

From this we deduce the following:

PROPOSITION 3.26: *Suppose $0 \neq z \in H_!^1(\overline{C}_{g,h}^{\mathrm{ord}}, \vartheta_p(k))$ is a cohomology class which transforms under $\langle a, a^{-1}\rangle$ via ε and under $\langle 1, b\rangle$ via $\alpha^{-1}(b)$. Then ε and α are characters $\bmod\, \ell^{max(g,h)}$. Furthermore, if $g \geq h$, then α is trivial on $1 + \ell^{\min(g,h)}\mathbb{Z}_\ell$. If $h \geq g$, then $\varepsilon\alpha^{-1}$ is trivial on $1 + \ell^{\min(g,h)}\mathbb{Z}_\ell$.*

Finally we need several facts concerning the correspondence U_ℓ.

46: *The correspondence U_ℓ on X_s is determined by its action on the ordinary locus. There applying the modular definition as, for example, in [18, Corollary 8.4] we see that:*

On $C_{e,0}$, $U_\ell = F_\ell +$ *maps into* $C_{e-1,u}$, $u \in (\mathbb{Z}/\ell\mathbb{Z})^\times$.
On $C_{0,e}$,$U_\ell = F_\ell^t =$ *Ver.*

Hence if ε is primitive there are two cases for U_ℓ.

47: *If ε is not primitive at ℓ, then by the modular definition of U_ℓ it is a projection operator from level $N = \ell^e$ to level ℓ^{e-1} (assuming $e > 1$).*

Hence for new representations if ε is not primitive then $U_\ell = 0$.

We are now in a position to analyze maximal ideals in the Hecke algebra such that the corresponding Galois representation is reducible. So let

$$\mathbb{T} = \mathbb{Z}\langle T_\ell\, ,\, \ell \nmid N; U_\ell\, ,\, \ell|N; \langle d\rangle, d \in (\mathbb{Z}/N\mathbb{Z})^\times\rangle$$

be the weight k Hecke algebra for $\Gamma_1(N)$ and suppose $\mathbf{m} \subseteq \mathbb{T}$ is an ideal of residue characteristic p with $\langle d\rangle - \varepsilon(d) \in \mathbf{m}\mathbb{T}'$ such that $\rho_{\mathbf{m}}$ is reducible, say $\rho_{\mathbf{m}} = \alpha \oplus \beta\chi^{k-1}$ with χ as usual the p-cyclotomic character. By the crystalline theory (Proposition 2.3) the characters α and β of $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ are unramified at p if $p > k$.

As previously we let $\overline{\vartheta}_p = \overline{\vartheta}_p(k)$ denote the torsion sheaf $\vartheta_p(k)/p\vartheta_p(k)$ on $X_1(N)$. Then the $\mathbb{T}[\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})]$-module $H^1(X, \overline{\vartheta}_p(k))^\vee[\mathbf{m}]$ has a Jordan–Hölder filtration with all constituents equal to constituents of $\rho_{\mathbf{m}}$. In particular there is a cohomology class $0 \neq z' \in H^1(X_{\overline{\eta}}, \overline{\vartheta}_p(k))[\mathbf{m}]$ such that the line it generates is $\mathbb{T}[\mathrm{Gal}\,(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)]$- invariant with $\langle d\rangle \cdot z' = \varepsilon(d) \cdot z'$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. By the Good Reduction Theorem of Katz–Mazur z' arises from a class $0 \neq z'' \in H^1(X_{\overline{s}}, \overline{\vartheta}_p(k))[\mathbf{m}]$ if $\mathrm{cond}_\ell(\varepsilon) > e/2$. But $H_!^1(X_{\overline{s}}^{\mathrm{ord}}, \overline{\vartheta}_p(k))$ surjects onto $H^1(X_{\overline{s}}, \overline{\vartheta}_p(k))$ by (42). Then the Hecke eigenvalues and Galois representations lift.

PROPOSITION 3.27: *Suppose $p > k$, ε is primitive at ℓ, and $\rho_{\mathbf{m}} = \alpha \oplus \beta\chi^{k-1}$. Then $\alpha\beta = \varepsilon$, α or β is unramified at ℓ, and $U_\ell =$ Trace of F_ℓ on the unramified part.*

Proof: Recall that H^1 corresponds to the dual of $\rho_{\mathbf{m}}$. By 3.26 and 46 we see that if ε is primitive then either

1. $g = e$, $h = 0$, α is trivial on inertia at ℓ, and $U_\ell = \alpha(\ell)$ or
2. $g = 0$, $h = e$, $\alpha = \varepsilon$ on inertia, and $U_\ell = \ell^{k-1}(\varepsilon\alpha^{-1})(\ell)$. This establishes the proposition. ∎

Note that if ε is primitive then any $\mathbf{m}$ is new. We now take up the case of non-primitive ε.

PROPOSITION 3.28: *Let the level $N = \ell^e N'$ with $(\ell, N') = 1$, $e > 1$. Suppose $p > k$, ε is not primitive at ℓ, and $\mathbf{m} \subseteq \mathbb{T}$ is new. Then $U_\ell \in \mathbf{m}$. Moreover if $e > \mathrm{cond}_\ell(\varepsilon) > e/2 \geq 1$ and $\rho_{\mathbf{m}} = \alpha \oplus \beta\chi^{k-1}$ occurs in cusp forms of type (N, k, ε), then $e \geq \mathrm{cond}_\ell(\alpha) + \mathrm{cond}_\ell(\beta)$.*

Proof: The assertion is just a restatement of the fact (47) and the discussion preceding Proposition 3.26. ∎

3.4 SEMI-STABLE REDUCTION AT ℓ: THE CASE $e = 1$. Let ℓ be a prime, $N = \ell N'$ with $(N', \ell) = 1$, and ε be a Dirichlet character defined on $(\mathbb{Z}/N'\mathbb{Z})^\times$. Denote by $\mathbb{T}$ the Hecke algebra for $S_k(\Gamma_0(N), \varepsilon)$—$\mathbb{T}$ is the $\mathbb{Z}$-subalgebra of $\mathrm{End}_{\mathbb{C}}\,(S_k(\Gamma_0(N), \varepsilon))$ generated by T_r, $r \nmid N$; U_r, $r | N$; and $\langle d \rangle$, $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. Suppose $\mathbf{m} \subseteq \mathbb{T}$ is a *new* maximal ideal of residue characteristic p with $p > k$, $(p, N) = 1$ such that the representation $\rho_{\mathbf{m}}$ is reducible. The crystalline theory (Proposition 2.3) shows that $\rho_{\mathbf{m}} = \alpha \oplus \beta\chi^{k-1}$ where χ is the p-cyclotomic character and $\varepsilon = \alpha\beta$. Moreover the characters α and β are unramified outside N; in particular they are unramified at p. As usual, for $r \nmid N$ the Eichler–Shimura relation gives:

$$T_r - (\alpha(r) + \beta(r) r^{k-1}) \in \mathbf{m}\mathbb{T}', \tag{48}$$

where $\mathbb{T}' = \mathbb{T} \otimes R$. Also the crystalline site gives us as usual that

$$T_p - (\alpha(p) + \beta(p) p^{k-1}) \in \mathbf{m}\mathbb{T}'. \tag{49}$$

The purpose of this subsection is to examine the action of U_ℓ in this case of semi-stable reduction. To study U_ℓ we consider the bad reduction at ℓ of $X_0(N)$. The special fiber of $X_0(N)/\mathbb{Z}_\ell$ consists of two copies of $X_0(N')/\overline{\mathbb{F}}_\ell$. Recall that ϕ: $E \longrightarrow Y_0(N)$ is the universal elliptic curve and $\vartheta_p(k) = \mathrm{Symm}^{k-2}(R^1\phi_*\mathbb{Z}_p)$. In general suppose $\mathcal{F}$ is a $\mathbb{Z}_p$ sheaf on $X_1(N)$ and τ: $(\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathcal{O}^\times$ is a character. Then we define $\mathcal{F}_\tau$ to be the largest subsheaf of $\mathcal{F} \otimes_{\mathbb{Z}_p} \mathcal{O}$ where $\langle d \rangle$ acts via $\tau(d)$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. Let π: $X_1(N) \longrightarrow X_0(N)$ be the natural projection. We then set $\mathcal{F}(\tau)$ equal to the sheaf $\pi_*(\mathcal{F}_\tau)$ on $X_0(N)$. Now we apply this construction to the sheaf $\vartheta_p(k)$. View the character ε as taking values in an extension $\mathcal{O}$ of $\mathbb{Z}_p$. Then we have the sheaf $\vartheta_p(k)(\varepsilon)$ on $X_0(N)$. Since ε has no component at ℓ, the sheaf $\vartheta_p(k)$ has "good reduction" at ℓ. So Picard-Lefschetz theory tells us that the action of inertia at ℓ is given by the local contribution at the double points Σ. Specifically we have the maps

$$(50)\qquad \begin{aligned} H^1_\Sigma(X_0(N)\otimes\overline{\mathbb{F}}_\ell,\vartheta_p(k)(\varepsilon)) &\longrightarrow H^1(X_0(N)\otimes\overline{\mathbb{F}}_\ell,\vartheta_p(k)(\varepsilon))\\ &\longrightarrow H^1(X_0(N)\otimes\overline{\mathbb{Q}}_\ell,\vartheta_p(k)(\varepsilon)). \end{aligned}$$

Let $I_\ell\subseteq \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ denote the inertia subgroup. Then for any $\sigma\in I_\ell$,

$$(51)\qquad \begin{aligned} &(\sigma-1)H^1(X_0(N)\otimes\overline{\mathbb{Q}}_\ell,\vartheta_p(k)(\varepsilon))\quad\text{is contained in}\\ &\mathrm{Image}\left(H^1_\Sigma(X_0(N)\otimes\overline{\mathbb{F}}_\ell,\vartheta_p(k)(\varepsilon))\longrightarrow H^1(X_0(N)\otimes\overline{\mathbb{Q}}_\ell,\vartheta_p(k)(\varepsilon))\right). \end{aligned}$$

PROPOSITION 3.29:

(1) *If* $\sigma\in I_\ell$, *then* $(\sigma-1)^2=0$ *on* $H^1(X_0(N)\otimes\overline{\mathbb{Q}}_\ell,\vartheta_p(k)(\varepsilon))$.

(2) *For* $\sigma\in I_\ell$, $(\sigma-1)H^1(X_0(N)\otimes\overline{\mathbb{Q}}_\ell,\vartheta_p(k)(\varepsilon))\subseteq H^1(X_0(N)\times\overline{\mathbb{Q}}_\ell,\vartheta_p(k)(\varepsilon))^{I_\ell}$.

Proof: Obviously (2) is simply a reformulation of (1). The assertion follows from SGA 7, exp. 15. There it is stated only for constant coefficients. However the proof applies verbatim to the case of locally constant coefficients. ∎

In particular inertia at ℓ acts unipotently so on a one-dimensional space it must operate trivially. It follows that the characters α and β are unramified at ℓ.

Now observe that

$$(52)\qquad \begin{aligned} &H^1_\Sigma(X_0(N)\otimes\overline{\mathbb{F}}_\ell,\vartheta_p(k)(\varepsilon))\\ &\quad\cong\sum_{x\in\Sigma}\left(\Gamma(\{\overline{x}\},\vartheta_p(k)(\varepsilon))\otimes_{\mathbb{Z}_p}H^1_{\{x\}}(X_0(N)\otimes\overline{\mathbb{F}}_\ell,\mathbb{Z}_p)\right). \end{aligned}$$

This decomposition makes it easy to see the action of the arithemetic Frobenius $\mathrm{Frob}_\ell=F_\ell$. The morphism Frob_ℓ acts on $\Gamma(\Sigma,\vartheta_p(k)(\varepsilon))$ as w_ℓ. To determine its action on $H^1_\Sigma(X_0(N)\otimes\overline{\mathbb{F}}_\ell,\mathbb{Z}_p)$, it suffices to consider the local picture of a double point. Let then X be two lines crossing at a point Q.Then

$$H^1_Q(X,\mathbb{Z}_p)=\mathrm{Coker}\,(\Gamma(X-Q),\mathbb{Z}_p)\leftarrow\Gamma(X,\mathbb{Z}_p)).$$

Now $\Gamma(X-Q),\mathbb{Z}_p)\cong\mathbb{Z}_p^2$ and $\Gamma(X,\mathbb{Z}_p)\cong\mathbb{Z}_p$. The quotient is one-dimensional, generated by one component or minus the other. So switching the components at the double point acts as -1. But w_ℓ acts on $X_0(N)\otimes\overline{\mathbb{F}}_\ell$ by switching the crossing components at each point of Σ. Hence w_ℓ acts on $H^1_\Sigma(X_0(N)\otimes\overline{\mathbb{F}}_\ell,\mathbb{Z}_p)$ as -1. Hence we have the theorem:

THEOREM 3.30:

(1) F_ℓ *acts on* $H^1_\Sigma(X_0(N)\otimes\overline{\mathbb{F}}_\ell,\vartheta_p(k)(\varepsilon))$ *as* $-w_\ell^{-1}$.

(2) $F_\ell=-w_\ell^{-1}$ *on* $(\sigma-1)H^1(X_0(N)\times\overline{\mathbb{Q}}_\ell,\vartheta_p(k)(\varepsilon))\subseteq H^1(X_0(N)\otimes\overline{\mathbb{Q}}_\ell,\vartheta_p(k)(\varepsilon))^{I_\ell}$ *for any* $\sigma\in I_\ell$.

We remark that for the case $e = 1$ we have that $U_\ell + w_\ell$ is a projector to lower level. Hence since $\mathbf{m}$ is new $U_\ell = -w_\ell$ on $H^1_{\mathrm{par}}(Y_0(N), \overline{\vartheta}_p(\varepsilon))[\mathbf{m}]$ where $\overline{\vartheta}_p = \vartheta_p(k)/p\vartheta_p(k)$. The eigenvalues of F_ℓ^{-1} acting on $H^1_{\mathrm{par}}(Y_0(N), \overline{\vartheta}_p(\varepsilon))[\mathbf{m}]$ are $\alpha(\ell)\ell^{k-1}$ and $\beta(\ell)$. Since $F_\ell^{-1} = U_\ell$ on $(\sigma - 1)H^1_{\mathrm{par}}(Y_0(N), \overline{\vartheta}_p(\varepsilon))[\mathbf{m}]$ we deduce:

Proposition 3.31: *The scalar by which U_ℓ acts on $H^1_{\mathrm{par}}(Y_0(N), \overline{\vartheta}_p(\varepsilon))[\mathbf{m}]$ is one of $\{\alpha(\ell), \beta(\ell)\ell^{k-1}\}$.*

Note that since $\rho_{\mathbf{m}} = \alpha \oplus \beta\chi^{k-1}$ we have that $\mathbf{k_m}$ is generated by the values of α and β. Thus $\mathbf{k_m} \subseteq R/\wp$.

Since $U_\ell^2 = w_\ell^2 = \ell^{k-2}\alpha\beta(\ell)$, there are two cases:

(53) $$\text{(i) If } U_\ell = \alpha(\ell), \text{then } \ell^{k-2} \equiv \alpha/\beta(\ell) \bmod p.$$

(54) $$\text{(ii) If } U_\ell = \beta(\ell)\ell^{k-1}, \text{ then } \ell^k \equiv \alpha/\beta(\ell) \bmod p.$$

Assume that $k > 2$ for simplicity and set

(55) $$\tilde{E}(\alpha, \beta; \ell)_1 = \alpha(\ell)E(\alpha, \beta) - w_\ell E(\alpha, \beta),$$

(56) $$\tilde{E}(\alpha, \beta; \ell)_2 = \beta(\ell)\ell^{k-1}E(\alpha, \beta) - w_\ell E(\alpha, \beta).$$

For $i = 1, 2$ the Eisenstein series $\tilde{E}(\alpha, \beta; \ell)_i$ are defined whenever

$$(\ell, N_{\alpha,\beta} \overset{\mathrm{def}}{=} \mathrm{cond}\,(\alpha)\mathrm{cond}\,(\beta)) = 1$$

and are modular forms of level $\ell N_{\alpha,\beta}$. They are eigenfunctions of the restricted Hecke algebra $\mathbb{T}_0$ with the same eigenvalues as $E(\alpha, \beta)$. Use the formulae $w_\ell + U_\ell = T_\ell$ and $U_\ell w_\ell = \ell^{k-1}\alpha\beta(\ell)$ to compute

(57) $$\text{(i) } U_\ell\tilde{E}(\alpha, \beta; \ell)_1 = \alpha(\ell)\tilde{E}(\alpha, \beta; \ell)_1,$$

(58) $$\text{(ii) } U_\ell\tilde{E}(\alpha, \beta; \ell)_2 = \beta(\ell)\ell^{k-1}\tilde{E}(\alpha, \beta; \ell)_2.$$

We now normalize the $\tilde{E}(\alpha, \beta; \ell)_i$, $i = 1, 2$. Observe that since $E(\alpha, \beta)$ has level $N_{\alpha,\beta}$ which is prime to ℓ, we have

(59) $$w_\ell E(\alpha, \beta) = \pi_\ell^* E(\alpha, \beta) = \ell^{k-1}V_\ell E(\alpha, \beta),$$

using the notation and results of Proposition 3.13 for the covering $\pi_\ell\colon X_1(N_{\alpha,\beta}\ell) \longrightarrow X_1(N_{\alpha,\beta})$. Hence the q-expansion of $w_\ell E(\alpha, \beta)$ at ∞ has $a_1 = 0$. We therefore set

(60) $$E(\alpha, \beta; \ell)_1 = \frac{1}{\alpha(\ell)}\tilde{E}(\alpha, \beta; \ell)_1,$$

(61) $$E(\alpha, \beta; \ell)_2 = \frac{1}{\beta(\ell)\ell^{k-1}}\tilde{E}(\alpha, \beta; \ell)_2.$$

Then for $i = 1, 2$ $E(\alpha, \beta; \ell)_i$ is normalized, i.e., its q-expansion at ∞ has $a_1 = 1$. It is an eigenfunction of the Hecke algebra $\mathbb{T}_0$ and also an eigenfunction of U_ℓ with eigenvalues given in equation (57). Hence we have constructed Eisenstein series which give packages of Hecke eigenvalues corresponding the the possibilities of Proposition 3.31.

3.5 REDUCIBLE $\mathbf{m} \subseteq \mathbb{T}$ ARE EISENSTEIN. We retain our previous notation; in particular $\mathbb{T}$ denotes the weight k Hecke algbra for $\Gamma_1(N)$, R is the finite extension of $\mathbb{Z}$ generated by the Nth roots of 1 together with the values of all Dirichlet characters of conductor dividing N, and $T' = \mathbb{T} \otimes R$. Suppose $\mathbf{m} \subseteq \mathbb{T}$ is a new reducible maximal ideal of residue characteristic $p > k$, $p \nmid N$ associated to a cusp form of type (k, N, ε). We suppose $\rho_{\mathbf{m}} = \overline{\alpha} \oplus \overline{\beta}\chi^{k-1}$ where χ is the p-cyclotomic character. By the crystalline theory $\overline{\alpha}$ and $\overline{\beta}$ are unramified at p. Let $\operatorname{cond}(\overline{\alpha})$, $\operatorname{cond}(\overline{\beta})$ be the Artin conductors of $\overline{\alpha}$, $\overline{\beta}$ (defined as products of local factors over all places except p). By Carayol [2] or Livné [13], $N_{\overline{\alpha},\overline{\beta}} = \operatorname{cond}(\overline{\alpha})\operatorname{cond}(\overline{\beta})|N$. Let $\mathbf{m}' \subseteq \mathbb{T}'$ be a maximal ideal of $\mathbb{T}'$ lying over $\mathbf{m}$, i.e. $\mathbf{m}' \cap \mathbb{T} = \mathbf{m}$. The ideal $\wp = \mathbf{m}' \cap R$ is prime and $k_{\mathbf{m}} \subseteq R/\wp$. The natural projection $\pi_\wp$: $R^\times \longrightarrow (R/\wp)^\times$ maps the Nth roots of unity $\mu_N \subseteq R^\times$ isomorphically onto $\mu_N \subseteq (R/\wp)^\times$. Set $i = \pi_\wp|\mu_N$. Define liftings of $\overline{\alpha}$ and $\overline{\beta}$ to Dirichlet characters by setting $\alpha = i^{-1} \circ \overline{\alpha}$, $\beta = i^{-1} \circ \overline{\beta}$. Then $\operatorname{cond}(\alpha) = \operatorname{cond}(\overline{\alpha})$ and $\operatorname{cond}(\beta) = \operatorname{cond}(\overline{\beta})$, so $N_{\alpha,\beta} = N_{\overline{\alpha},\overline{\beta}}|N$.

The eigenvalues of $T_\ell \bmod \mathbf{m}$, $\ell \nmid N$, and $\langle d\rangle \bmod \mathbf{m}$ are known (cf. Proposition 3.25); namely $I(\alpha, \beta)_0 \subseteq \mathbf{m}_0$. Suppose ℓ is a prime and $N = \ell^e N'$ with $e \geq 1$ and $(N', \ell) = 1$. Set $\mathbb{T}' = \mathbb{T} \otimes \mathbb{Z}[\alpha, \beta]$, $\mathbb{T}'_0 = \mathbb{T} \otimes \mathbb{Z}[\alpha, \beta]$, $\mathbf{m}'_0 = \mathbf{m}_0\mathbb{T}'_0$, and $\mathbf{m}' = \mathbf{m}\mathbb{T}'$. The previous two sections determine the eigenvalues of $U_\ell \bmod \mathbf{m}$, which we restate here for convenience.

THEOREM 3.32:

(1) *If ε is primitive at ℓ, then $U_\ell - (\alpha(\ell) + \beta(\ell)\ell^{k-1}) \in \mathbf{m}$ and one of α, β is unramified at ℓ.*

(2) *If ε is not primitive at ℓ and $e > 1$, then $U_\ell \in \mathbf{m}$. If $\operatorname{cond}_\ell(\varepsilon) > e/2 \geq 1$, then $e \geq \operatorname{cond}_\ell(\alpha) + \operatorname{cond}_\ell(\beta)$.*

(3) *If ε is not primitive at ℓ and $e = 1$, then α and β are unramified at ℓ and either $U_\ell - \alpha(\ell) \in \mathbf{m}$ or $U_\ell - \beta(\ell)\ell^{k-1} \in \mathbf{m}$.*

We now show that in any of the above cases we can construct an Eisenstein series with this package of eigenvalues. The Eisenstein series we take will be a modification of the Eisenstein series $E(\alpha,\beta)$. Define operators Λ_ℓ for $\ell|N$ according to the cases of Theorem 3.32 as follows:

CASE 1: Suppose ε is primitive at ℓ. Then set $\Lambda_\ell = 1$. Note that in this case $\mathrm{cond}_\ell(\alpha) + \mathrm{cond}_\ell(\beta) = e$.

CASE 2: Suppose ε is not primitive at ℓ and $e > 1$. If both α and β are ramified at ℓ, set $\Lambda_\ell = 1$. If one of α or β is unramified at ℓ, then $\mathrm{cond}_\ell(\alpha)+\mathrm{cond}_\ell(\beta) < e$. In this case set $\Lambda_\ell = 1 - V_\ell U_\ell$.

CASE 3: Suppose ε is not primitive at ℓ and $e = 1$. If $U_\ell - \alpha(\ell) \in \mathbf{m}$, set

$$\Lambda_\ell = \frac{\alpha(\ell) + w_\ell}{\alpha(\ell)}.$$

If $U_\ell - \beta(\ell)\ell^{k-1} \in \mathbf{m}$, set

$$\Lambda_\ell = \frac{\beta(\ell)\ell^{k-1} - w_\ell}{\beta(\ell)\ell^{k-1}}.$$

Now consider the Eisenstein series

$$\tilde{E} = (\Pi_{\ell|N}\Lambda_\ell)E(\alpha,\beta), \tag{62}$$

and let E be the scalar multiple of $\tilde{E}$ which is normalized. If E has q-expansion at ∞ given by $E(\infty)(q) = \sum_{n=0}^{\infty} a_n(E)q^n$ then $a_1(E) = 1$, $T_\ell E = a_\ell(E)E$ for primes $\ell \nmid N$, $U_\ell E = a_\ell(E)E$ for primes $\ell|N$, and $\langle d\rangle E = \varepsilon(d)E$. Moreover, we have that

$$\langle T_\ell - a_\ell(E),\ \ell\nmid N; U_\ell - a_\ell(E),\ \ell|N; \langle d\rangle\rangle \subseteq \mathbf{m}'.$$

Let f be the normalized cusp form with coefficients in $\mathbf{k} = \mathbb{T}/\mathbf{m}$ associated to $\mathbf{m}$. Then $f - E$ has q-expansion at ∞ equal to a constant. We deduce cases where this constant must be 0, and hence where $E \bmod \wp$ is a cusp form, from the following lemma.

LEMMA 3.33: *Suppose $f = 1$ is the q-expansion about a multiplicative cusp of a modular form* $\bmod p$ *of weight k on $\Gamma_1(N)$, $(N,p) = 1$. Then $p-1$ divides k.*

Proof: There are two natural "degeneracy" maps for a prime ℓ, namely B_1, B_ℓ: $Y_0(N\ell) \longrightarrow Y_0(N)$. The map B_1 is defined in terms of moduli as $(E,x) \mapsto (E,\ell x)$, where x is a point of the elliptic curve E of exact order $N\ell$. This is the usual forgetful map and corresponds to the map on the Poincaré upper half plane given by $z \mapsto z$. The map B_ℓ is defined in terms of moduli as $(E,x) \mapsto (E/\langle \ell x\rangle, \overline{x})$ and corresponds to the map $z \mapsto \ell z$ on the Poincaré upper half plane.

There are two cusps of $Y_0(N\ell)$ lying above the multiplictive cusp of $Y_0(N)$ about which f has the q-expansion equal to the constant 1. One of the cusps will be étale at ℓ and the other will be multiplicative at ℓ. It is easily seen that $B_1^*(f)$ has q-expansion about both these cusps of $Y_0(N\ell)$ equal to 1, whereas $B_\ell^*(f)$ has q-expansion about the ℓ-multiplicative cusp of ℓ^k and q-expansion about the ℓ-étale cusp equal to 1.

Under the assumption $\ell \equiv 1 \bmod N$ we then must have $B_\ell^*(f) = B_1^*(f) = \ell^k B_1^*(f)$, so $\ell^k \equiv 1 \bmod p$. Since this holds for all $\ell \equiv 1 \bmod N$ and $(\ell, N) = 1$, this means that $(p-1)|k$. ∎

The Eisenstein series classically denoted E_{p-1} of weight $p-1$ on $SL(2,\mathbb{Z})$ has q-expansion congruent to 1 mod$p$, showing that the q-expansion about a multiplicative cusp $f = 1$ does indeed arise when the weight k is divisible by $p-1$.

We now compute the constant term of E. For primes ℓ in <u>Case 3</u> above, we use the following.

PROPOSITION 3.34: *Suppose $N = \ell N'$ with $(N', \ell) = 1$. Let g be a modular form of weight k and level N', so $w_\ell g$ has level N.*

1. *The modular form $(\alpha(\ell) - w_\ell)g$ has constant term $\alpha(\ell)(1-\ell^{k-1})a_0(g;c)$ at an ℓ-multiplicative cusp c, $\alpha(\ell)(1-1/\ell)a_0(g;c)$ at an ℓ-étale cusp c.*
2. *The modular form $(\beta(\ell)\ell^{k-1} - w_\ell)g$ has constant term 0 at an ℓ-multiplicative cusp, $(\ell^{k-1}\beta(\ell) - \alpha(\ell)/\ell)a_0(g;c)$ at an ℓ-étale cusp c.*

Analyzing the effect of primes ℓ in <u>Case 2</u> is more laborious. We will proceed via a series of propositions.

PROPOSITION 3.35: *Let $f \in M_k(\Gamma_1(N))$ with $N = \ell^a N'$ for ℓ a prime, $a \geq 1$, and $(\ell, N') = 1$. Suppose R is a Dedekind domain such that $a_0(f) \in R[\text{orientedcuspsof} X_1(N)]$ is primitive. Assume that there exists an integer $0 \leq b \leq a$ such that $a_0(f;c) = 0$ if c is an oriented cusp of $X_1(N)$ with $N_e(c) = \ell^b N_e'(c)$ with $(\ell, N_e'(c)) = 1$. Then for any unit $\lambda \in R^\times$, $a_0((\pi_1^* - \lambda\pi_\ell^*)f) \in R[\text{oriented cusps of } X_1(N\ell)]$ is primitive.*

Proof: For a cusp $\tilde{c}$ of $X_1(N\ell)$, routine computation shows that:

If $\ell | N_e(\tilde{c})$, $N_e(\pi_1\tilde{c}) = N_e(\tilde{c})/\ell$. If $\ell \nmid N_e(\tilde{c})$, $N_e(\pi_1\tilde{c}) = N_e(\tilde{c})$.
If $\ell | N_m(\tilde{c})$, $N_e(\pi_\ell\tilde{c}) = N_e(\tilde{c})$. If $\ell \nmid N_m(\tilde{c})$, $N_e(\pi_\ell\tilde{c}) = N_e(\tilde{c})/\ell$.

Let $\wp$ be a prime of R and suppose $a_0(F) \neq 0$. By hypothesis one of the following two cases holds.

Case 1. There is an oriented cusp $c = \langle \zeta, i/N_e(c) \rangle$ with

$$a_0(f;c) \not\equiv 0 \bmod \wp, \quad v_\ell(N_m(c)) \geq 1,$$

and $a_0(f;c') \equiv 0 \bmod \wp$ for all c' with

$$v_\ell(N_e(c)) - v_\ell(N_e(c')) = 1.$$

Case 2. There is an oriented cusp $c = \langle \zeta, i/N_e(c) \rangle$ with

$$a_0(f;c) \not\equiv 0 \bmod \wp, \quad v_\ell(N_m(c)) \geq 1,$$

and $a_0(f;c') \equiv 0 \bmod \wp$ for all c' with

$$v_\ell(N_e(c')) - v_\ell(N_e(c)) = 1.$$

In Case 1, consider the cusp $\tilde{c} = \langle \zeta^{1/\ell}, i/N_e(c) \rangle$. Then $\pi_\ell(\tilde{c}) = c$ and $N_e(\pi_1 \tilde{c}) = N_e(\tilde{c})/\ell = N_e(c)/\ell$. Hence

$$\begin{aligned} a_0((\pi_1^* - \lambda \pi_\ell^*)f; \tilde{c}) &= a_0(f;\tilde{c}) - \lambda a_0(\pi_\ell^* f; \tilde{c}) \\ &= a_0(f;\tilde{c}) - \lambda a_0(f; (\pi_\ell)_* \tilde{c}) \equiv -\lambda a_0(f;c) \bmod \wp. \end{aligned}$$

In Case 2, consider the cusp $\tilde{c} = \langle \zeta^{1/\ell}, i/\ell N_e(c) \rangle$. Then $\pi_1(\tilde{c}) = c$ and $N_e(\pi_\ell \tilde{c}) = N_e(\tilde{c}) = N_e(c) + 1$. Hence

$$\begin{aligned} a_0((\pi_1^* - \lambda \pi_\ell^*)f; \tilde{c}) &= a_0(f;c) - \lambda a_0(\pi_\ell^* f; \tilde{c}) \\ &= a_0(f;c) - \lambda a_0(f; (\pi_\ell)_* \tilde{c}) \equiv a_0(f;c) \bmod \wp. \end{aligned}$$

So in either case $a_0((\pi_1^* - \lambda \pi_\ell^*)f) \not\equiv 0 \bmod \wp$ for each prime $\wp$ and lattice element

$$a_0((\pi_1^* - \lambda \pi_\ell^*)f) \in R[\textit{oriented cusps of } X_1(N\ell)]$$

is primitive. ∎

PROPOSITION 3.36: *Let α and β be Dirichlet characters of conductors* cond (α) *and* cond (β) *respectively. Suppose $N_{\alpha,\beta} = \mathrm{cond}\,(\alpha)\mathrm{cond}\,(\beta) = \ell^a N'_{\alpha,\beta}$ for ℓ a prime, $a \geq 1$, and $(\ell, N'_{\alpha,\beta}) = 1$. If α is unramified at ℓ, then $a_0(E(\alpha,\beta);c) = 0$ for any oriented cusp c of $X_1(N_{\alpha,\beta})$ with $v_\ell(N_m(c)) = 0$. If β is unramified at ℓ, then $a_0(E(\alpha,\beta);c) = 0$ for any oriented cusp c of $X_1(N_{\alpha,\beta})$ with $v_\ell(N_e(c)) = 0$.*

Proof: We recall the notation of Definition 3.16 . We factor $N_{\alpha,\beta} = S'T'$ where $\mathrm{cond}_\ell(\beta) > \mathrm{cond}_\ell(\alpha)$ if $\ell|S'$ and $\mathrm{cond}_\ell(\alpha) \geq \mathrm{cond}_\ell(\beta)$ if $\ell|T'$. Moreover we set $S = \mathrm{cond}(\beta_{S'})$ and $T = \mathrm{cond}(\alpha_{T'})$. Then $w_{\zeta_S} E(\alpha\beta^{-1}, 1)$ is a modular form of level ST. Also from Proposition 3.11 we have

(i) If $\ell|S$, $a_0(w_{\zeta_S} E(\alpha\beta^{-1}, 1); c') = 0$ for any oriented cusp c' of $X_1(ST)$ with $v_\ell(N_m(c')) = 0$.

(ii) If $\ell|T$, $a_0(w_{\zeta_S} E(\alpha\beta^{-1}, 1); c') = 0$ for any oriented cusp c' of $X_1(ST)$ with $v_\ell(N_e(c')) = 0$.

Suppose in general that E is a Tate curve and x is a point of E of exact order N. Let $\lambda: (E, x) \longrightarrow (E_\lambda, x_\lambda)$ be an isogeny of degree prime to N. Then x_λ is of exact order N and $N_e(x_\lambda) = N_e(x)$. Now

$$E(\alpha, \beta) = \frac{1}{\lambda(\alpha\beta^{-1}, S)} w_{\zeta_S} E(\alpha\beta^{-1}, 1) \otimes \alpha_{S'}\beta_{T'}$$

and $\ell \nmid \mathrm{cond}(\alpha_{S'})\mathrm{cond}(\beta_{T'})$ by the hypothesis that α or β is unramified at ℓ. Hence from the definition of twisting (3.16) we see that for any oriented cusp c on $X_1(N_{\alpha,\beta})$, $a_0(E(\alpha, \beta); c)$ is a linear combination of terms of the form $a_0(w_{\zeta_S} E(\alpha\beta^{-1}, 1); c')$ with c' a cusp of $X_1(ST)$ satisfying $v_\ell(N_e(c')) = v_\ell(N_e(c))$. Hence the proposition follows from (i), (ii) above. ∎

PROPOSITION 3.37: *Let α and β be Dirichlet characters of conductors* $\mathrm{cond}(\alpha)$, $\mathrm{cond}(\beta)$ *respectively. Suppose that the prime ℓ divides* $\mathrm{cond}(\alpha)\mathrm{cond}(\beta) = N_{\alpha,\beta}$ *and one of α or β is unramified at ℓ. Then $a_0(1 - V_\ell U_\ell)E(\alpha, \beta)) = L(1-k, \alpha^{-1}\beta) \cdot v$, where v is a primitive vector in the lattice $\mathbb{Z}[1/N_{\alpha,\beta}, \alpha, \beta]$ [oriented cusps of $X_1(N_{\alpha,\beta}\ell)$].*

Proof: By Theorem 3.17,

$$U_\ell E(\alpha, \beta) = \lambda_\ell E(\alpha, \beta) \quad \text{with } \lambda_\ell = \alpha(\ell) + \beta(\ell)\ell^{k-1}.$$

As $\alpha(\ell) = 0$ or $\beta(\ell) = 0$, it follows that λ_ℓ is a unit in the ring $R = \mathbb{Z}[1/N_{\alpha,\beta}, \alpha, \beta]$. Now

$$(1 - V_\ell U_\ell)E(\alpha, \beta) = (\pi_1^* - \lambda_\ell \ell^{-(k-1)} \pi_\ell^*)E(\alpha, \beta),$$

cf. Proposition 3.13 . The lattice element

$$(1/L(1-k, \alpha^{-1}\beta))a_0(E(\alpha, \beta)) \in R[\text{oriented cusps of } X_1(N_{\alpha,\beta})]$$

is primitive by Theorem 3.20 . Hence from Propositions 3.35 and 3.36 we deduce that

$$a_0((1 - V_\ell U_\ell)E(\alpha,\beta)) = L(1-k,\alpha^{-1}\beta)\cdot v$$

where v is a primitive vector in the lattice R[*oriented cusps of* $X_1(N_{\alpha,\beta}\ell)$]. ∎

Putting all this together we can therefore settle the following case of Conjecture 3.22:

THEOREM 3.38: *Let $\mathbb{T}$ be the weight k Hecke algebra for $\Gamma_1(N)$. Suppose $\mathbf{m} \subseteq \mathbb{T}$ is a new reducible maximal ideal of residue characteristic p with $p > k+1$, $p \not| N$. Then $\mathbf{m}$ is Eisenstein (of level N).*

Proof: Suppose $\rho_{\mathbf{m}} = \overline{\alpha} \oplus \overline{\beta}\chi^{k-1}$, where χ is the p-cyclotomic character. Construct Dirichlet characters α, β: $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow R^\times \subseteq \mathbb{C}^\times$ as in the beginning of Section 3.5 such that $\alpha \bmod \wp = \overline{\alpha}$, $\beta \bmod \wp = \overline{\beta}$ for a fixed prime ideal $\wp$ of R lying above p. Let E be the Eisenstein series obtained by normalizing $\tilde{E}$ as in equation (62). Then if $E = \sum_{n=0}^{\infty} a_n q^n$, we have $T_\ell E = a_\ell E$ for $\ell \not| N$, $U_\ell E = a_\ell E$ for $\ell | N$, and $\langle d\rangle E = \varepsilon(d)E$. For a prime $\ell | N$, set

$$\lambda_\ell = \ell - 1 \text{ if } U_\ell \bmod \mathbf{m} = \overline{\alpha}(\ell),$$
$$\lambda_\ell = \ell^k - (\alpha/\beta)(\ell) \text{ if } U_\ell \bmod \mathbf{m} = \overline{\beta}(\ell)\ell^{k-1}.$$

By Propositions 3.34–3.37 we have that

$$a_0(E) = L(1-k,\alpha^{-1}\beta)\Pi_{\ell || N}\lambda_\ell \cdot v,$$

where v is a primitive vector in $R[1/N]$[*oriented cusps of* $X_1(N)$]. Hence

$$\mathbf{m}' = \langle T_\ell - a_\ell \text{ for primes } \ell \not| N,\, U_\ell - a_\ell \text{ for primes } \ell | N; \langle d\rangle - \varepsilon(d), \wp\rangle \subseteq \mathbb{T}'$$

lies above $\mathbf{m} \subseteq \mathbb{T}$ with the prime $\wp$ dividing $L(1-k,\alpha^{-1}\beta)\Pi_{\ell||N}\lambda_\ell$. Hence $\mathbf{m}$ is Eisenstein. ∎

4. Multiplicity One for Eisenstein ideals

Let N be a positive integer and denote by $\mathbb{T}$ the weight k Hecke algebra for $\Gamma_0(N)$, $k \geq 2$. So $\mathbb{T}$ is generated over $\mathbb{Z}$ by T_ℓ, $\ell \not| N$; U_ℓ, $\ell | N$. Suppose $\mathbf{m} \subseteq \mathbb{T}$ is a maximal ideal with residue field $\mathbf{k} = \mathbb{T}/\mathbf{m}$ of finite characteristic $p > k$ prime to N. We retain our previous notation that ϕ: $E \longrightarrow Y_0(N)$ is the universal elliptic curve, $\vartheta_p = \mathrm{Symm}^{k-2}(R^1\phi_*\mathbb{Z}_p)$, and $\overline{\vartheta}_p = \vartheta_p/p\vartheta_p$. We have seen in Theorem 2.1 that if $\rho_{\mathbf{m}}$ is irreducible then $\dim_{\mathbf{k}} H^1(Y_0(N)_{\overline{\mathbf{Q}}}, \overline{\vartheta}_p)[\mathbf{m}] = 2$. In fact in this case $H^1(Y_0(N)_{\overline{\mathbf{Q}}}, \overline{\vartheta}_p)[\mathbf{m}]$ is isomorphic to the $\mathbf{k}[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module corresponding to $\rho_{\mathbf{m}}^\vee$. In this chapter we study the analogous question when $\rho_{\mathbf{m}}$ is reducible. If $\rho_{\mathbf{m}} = \alpha \oplus \beta$ for characters α and β, then all constituents of the $\mathbf{k}[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module $H^1(Y_0(N)_{\overline{\mathbf{Q}}}, \overline{\vartheta}_p)[\mathbf{m}]$ are isomorphic to α^{-1} or β^{-1}. More generally this is true for the $\mathbf{m}$-primary submodule $H^1_{\mathrm{par}}(Y_0(N)_{\overline{\mathbf{Q}}}, \overline{\vartheta}_p)\langle\mathbf{m}\rangle$ of $H^1_{\mathrm{par}}(Y_0(N)_{\overline{\mathbf{Q}}}, \overline{\vartheta}_p)$. We shall examine the case when N is a prime. If $\rho_{\mathbf{m}} = \alpha \oplus \beta$ then we know that $\{\alpha, \beta\} = \{\chi_0 = \text{trivial}, \chi^{k-1}\}$, where χ is the p-cyclotomic character. Let us assume that $\alpha = \chi_0$ and $\beta = \chi^{k-1}$.

We first study the possible extensions annihilated by p

$$0 \longrightarrow \chi_1 \longrightarrow \bullet \longrightarrow \chi_2 \longrightarrow 0,$$

where χ_1, $\chi_2 = \alpha$ or β, which satisfy the local requirements necessary to be a subquotient of the new part of $H^1_{\mathrm{par}}(Y_0(N)_{\overline{\mathbf{Q}}}, \overline{\vartheta}_p)^\vee$. The local requirements are:

1. At a prime $\ell \neq p, N$ the extension $0 \longrightarrow \chi_1 \longrightarrow \bullet \longrightarrow \chi_2 \longrightarrow 0$ must be unramified.
2. The extension $0 \longrightarrow \chi_1 \longrightarrow \bullet \longrightarrow \chi_2 \longrightarrow 0$ of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-modules is crystalline.
3. By Theorem 3.30, a Frobenius element F_N acts on

$$\{(\sigma - 1)x \mid x \in \bullet,\ \sigma \text{ an element of inertia at } N\}$$

(i.e., the smallest submodule of $\bullet$ such that the quotient is unramified) as $-Nw_N$. This shift occurs because subobjects correspond to quotients under duality. In general therefore if $0 \longrightarrow \chi_1 \longrightarrow \bullet \longrightarrow \chi_2 \longrightarrow 0$ ramifies at N, then

$$\chi_1(N) \equiv -N^{-1}w_N \bmod \mathbf{m} \text{ and } \chi_2(N) \equiv -w_N \bmod \mathbf{m}.$$

On the N-new part $w_N = -U_N$ since $T_N = w_N + U_N$ is a projection to the N-old part. Hence we can express the above equations in terms of U_N as

$$\chi_1(N) \equiv N^{-1}U_N \bmod \mathbf{m} \quad \text{and} \quad \chi_2(N) \equiv U_N \bmod \mathbf{m}.$$

A key ingredient in our study of such extensions will be the computation of certain Ext groups:

PROPOSITION 4.1: *Denote by* Ext$'$ *crystalline extensions annihilated by* p. *Let* $0 \le i, j \le p-2$ *and let* $\mathbf{F}_p(i)$ *be the Tate twist corresponding to* χ^i.

1. *Over* $\mathbb{Q}_p$,

$$\operatorname{Ext}'_{\mathbb{F}_p}(\mathbb{F}_p(i), \mathbb{F}_p(j)) = 0 \qquad \text{if } i - j > 0$$
$$\operatorname{Ext}'_{\mathbb{F}_p}(\mathbb{F}_p(i), \mathbb{F}_p(j)) \cong \mathbb{F}_p \qquad \text{if } i - j \le 0.$$

2. *Over* $\mathbb{Q}_p^{\mathrm{unr}}$,

$$\operatorname{Ext}'_{\mathbb{F}_p}(\mathbb{F}_p(i), \mathbb{F}_p(j)) = 0 \qquad \text{if } i - j \ge 0$$
$$\operatorname{Ext}'_{\mathbb{F}_p}(\mathbb{F}_p(i), \mathbb{F}_p(j)) \cong \mathbb{F}_p \qquad \text{if } i - j < 0.$$

Proof: Let $\mathbb{F}_p\{i\}$ be the crystalline object corresponding (in a contravariant manner) to the Tate twist $\mathbb{F}_p(i)$. The underlying module of $\mathbb{F}_p\{i\}$ is just $\mathbb{F}_p$, with filtration given by $F^i(\mathbb{F}_p\{i\}) = \mathbb{F}_p\{i\}$, $F^{i+1}(\mathbb{F}_p\{i\}) = (0)$, and $\varphi^i(1) = 1$. We then have

$$\operatorname{Ext}'_{\mathbb{F}_p}(\mathbb{F}_p(i), \mathbb{F}_p(j)) \cong \operatorname{Hom}(\mathbb{F}_p\{-j\}, \mathbb{F}_p\{-i\})/(1-\varphi^0)F^0 \operatorname{Hom}(\mathbb{F}_p\{-j\}, \mathbb{F}_p\{-i\}).$$

Moreover $\operatorname{Hom}(\mathbb{F}_p\{i\}, \mathbb{F}_p\{j\}) \cong \mathbb{F}_p\{i-j\}$. If $M = \mathbb{F}_p\{k\}$, then for $k < 0$ we have $F^0 M = 0$ while for $k > 0$ we have $F^0(M) = M$, $\varphi^0 = 0$. Over $\mathbb{Q}_p^{\mathrm{unr}}$ the situation is similar except that $1 - \varphi^0$ becomes surjective. ∎

We proceed to analyze the the various possibilities for $0 \longrightarrow \chi_1 \longrightarrow \bullet \longrightarrow \chi_2 \longrightarrow 0$:

<u>Case 1</u>. $0 \longrightarrow \alpha \longrightarrow \bullet \longrightarrow \alpha \longrightarrow 0$

There are no nontrivial crystalline extensions of $\mathbb{F}_p$ by $\mathbb{F}_p$ annihilated by p (cf. Proposition 4.1). Equivalently if $I \subset \operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ denotes the inertia subgroup then such an extension of I-modules is split. By 3) of the local requirements above we see that if the extension is ramified at N then $N \equiv U_N \bmod \mathbf{m}$ and $U_N \equiv 1 \bmod \mathbf{m}$, so $N \equiv 1 \bmod p$.

<u>Case 2</u>. $0 \longrightarrow \beta \longrightarrow \bullet \longrightarrow \beta \longrightarrow 0$

Again let $I \subset \operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ denote the inertia subgroup. The extension of I-modules is split at p since the crystalline $\operatorname{Ext}'_{\mathbf{F}_p}(\mathbb{F}_p(k-1), \mathbb{F}_p(k-1)) = 0$ over $\mathbb{Q}_p^{\mathrm{unr}}$. Deduce from 3) of the local requirements that if the extension is ramified at N then $N^k \equiv U_N \bmod \mathbf{m}$ and $U_N \equiv N^{k-1} \bmod \mathbf{m}$, so $N \equiv 1 \bmod p$.

<u>Case 3</u>. $0 \longrightarrow \alpha \longrightarrow \bullet \longrightarrow \beta \longrightarrow 0$

As before the extension of I-modules is split at p since the crystalline $\mathrm{Ext}'_{\mathbb{F}_p}(\mathbb{F}_p(k-1), \mathbb{F}_p) = 0$ over $\mathbb{Q}_p^{\mathrm{unr}}$. We see from (3) above that if the extension is ramified at N then $U_N \equiv N \bmod \mathbf{m}$ and $N^{k-1} \equiv U_N \bmod \mathbf{m}$, so $N^{k-2} \equiv 1 \bmod p$.

$$\underline{\text{Case 4}}. \quad 0 \longrightarrow \beta \longrightarrow \bullet \longrightarrow \alpha \longrightarrow 0$$

The crystalline $\mathrm{Ext}'_{\mathbb{F}_p}(\mathbb{F}_p, \mathbb{F}_p(k-1)) \cong \mathbb{F}_p$ over $\mathbb{Q}_p^{\mathrm{unr}}$. Locally at N we see from 3) above that if the extension is ramified at N then $N^k \equiv U_N \equiv 1 \bmod \mathbf{m}$, so $N^k \equiv 1 \bmod p$.

The global implications of these local results are as follows:

PROPOSITION 4.2: *For $\chi_1, \chi_2 \in \{\alpha, \beta\}$, denote by $\widetilde{\mathrm{Ext}}(\chi_1, \chi_2)$ the group of extensions of χ_2 by χ_1 satisfying the local requirements above.*

Let C denote the class group of $\mathbb{Q}(\zeta_p)$ and set $A = C/pC$. For a character ψ of $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$, A^ψ is the subgroup of A on which $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ acts via ψ. All such characters ψ are powers of the Teichmüller character ω.

(1) *If $N \not\equiv 1 \bmod p$, then $\widetilde{\mathrm{Ext}}(\alpha, \alpha) = \widetilde{\mathrm{Ext}}(\beta, \beta) = 0$.*
(2) $\dim_{\mathbb{F}_p}(\widetilde{\mathrm{Ext}}(\beta, \alpha)) \leq 1 + \mathrm{ord}_p |A^\psi|$, *where $\psi = \omega^{k-1}$, with equality only if $N \equiv U_N \bmod \mathbf{m}$ and $N^{k-2} \equiv 1 \bmod p$.*
(3) *If $U_N \not\equiv 1 \bmod \mathbf{m}$, then*

$$\dim_{\mathbb{F}_p}(\widetilde{\mathrm{Ext}}(\alpha, \beta)) \leq 1 + \mathop{\mathrm{ord}}_p |A^\psi|, \text{ where } \psi = \omega^{1-k}.$$

Proof: There is a natural equivalence of data between extensions

$$E: 0 \longrightarrow \chi^i \longrightarrow \bullet \longrightarrow \chi^j \longrightarrow 0$$

of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules annihilated by p and $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$-equivariant homomorphisms

$$\rho = \rho(E) \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_p)) \longrightarrow \mathrm{Hom}(\mathbb{F}_p(j), \mathbb{F}_p(i)) \cong \mathbb{F}_p(i-j).$$

The splitting field over $\mathbb{Q}(\zeta_p)$ of an extension E is the extension $\mathbb{Q}(\zeta_p)(\rho(E))$ cut out by $\rho(E)$. We first consider extensions E in $\widetilde{\mathrm{Ext}}(\alpha, \alpha)$ and $\widetilde{\mathrm{Ext}}(\beta, \beta)$. If $N \not\equiv 1 \bmod p$, then by Case 1 and Case 2 the extension $\mathbb{Q}(\zeta_p)(\rho(E))$ is an everywhere unramified extension of $\mathbb{Q}(\zeta_p)$ on which $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ acts trivially. But the only such are trivial since the ideal class group of $\mathbb{Q}$ is trivial, proving (1).

Now consider extensions E in $\widetilde{\mathrm{Ext}}(\beta, \alpha)$). By Case 3, the extension $\mathbb{Q}(\zeta_p)(\rho(E))/\mathbb{Q}(\zeta_p)$ is unramified outside N and ramified at N only if $N \equiv$

$1 \bmod p$ and $U_N \equiv 1 \bmod \mathbf{m}$. The possible E unramified at N correspond to $A^{\omega^{1-k}}$ by classfield theory. Allowing ramification above N adds at most 1 more dimension since the tame inertia group is cyclic, thereby proving (2).

Lastly consider extensions E in $\widetilde{\mathrm{Ext}}\,(\alpha, \beta))$. By Case 4, such extensions are unramified outside pN. Moreover if $N^{k-2} \not\equiv 1 \bmod p$ or $U_N \not\equiv 1 \bmod \mathbf{m}$, then the extension E is unramified at N. In this case, restriction to the intertia group gives a map

$$\widetilde{\mathrm{Ext}}\,(\alpha, \beta)) \longrightarrow \mathrm{Ext}'_{\mathbb{F}_p}(\mathbb{F}_p, \mathbb{F}_p(k-1)) \cong \mathbb{F}_p,$$

with Ext$'$ denoting the crystalline extensions over $\mathbb{Q}_p^{\mathrm{unr}}$ annihilated by p as in Proposition 4.1. The kernel of this map is the everywhere unramified extensions, i.e., $A^{\omega^{k-1}}$. This then proves (3). ∎

As in Proposition 4.2 above, let C denote the class group of $\mathbb{Q}(\zeta_p)$, $A = C/C^p$, and A^ψ denote the subgroup of A on which Gal$(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ acts via the character ψ. Any such ψ is a power of the Teichmüller character ω. The theorem of Herbrand–Ribet says that for $\psi = \omega^i$ with i odd, $2 < i \leq p-3$, $A^\psi = 0$ if and only if $p \nmid L(0, \psi^{-1})$. Hence the following proposition then follows from Proposition 4.2.

Proposition 4.3: *Let $p > k$ be coprime to the prime number N. Suppose $N \not\equiv 1 \bmod p$ and $U_N \not\equiv 1 \bmod \mathbf{m}$. Moreover if $2 < k < p-1$, suppose $p \nmid L(0, \omega^{k-1})L(0, \omega^{p-k})$. Then*

$$\widetilde{\mathrm{Ext}}\,(\alpha, \alpha) = \widetilde{\mathrm{Ext}}\,(\beta, \beta) = 0 \text{ and } \dim_{\mathbb{F}_p}(\widetilde{\mathrm{Ext}}\,(\beta, \alpha)) \leq 1, \dim_{\mathbb{F}_p}(\widetilde{\mathrm{Ext}}\,(\alpha, \beta)) \leq 1.$$

Next suppose

$$\widetilde{\mathrm{Ext}}\,(\alpha, \alpha) = \widetilde{\mathrm{Ext}}\,(\beta, \beta) = 0 \text{ and } \dim_{\mathbb{F}_p} \widetilde{\mathrm{Ext}}\,(\beta, \alpha)) \leq 1, \dim_{\mathbb{F}_p} \widetilde{\mathrm{Ext}}\,(\alpha, \beta) \leq 1.$$

Set $M_1 = \alpha$ and $N_1 = \beta$. Inductively suppose that a $\mathbf{k}[\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})]$-module N_{n-1} has been constructed , $n \geq 2$, satisfying the necessary local requirements 1,2,3 above to be a subquotient of $H^1_{\mathrm{par}}(Y_0(N)_{\overline{\mathbb{Q}}}, \overline{\vartheta}_p)^\vee$. Then define M_n to be a nontrivial extension

$$0 \longrightarrow \alpha \longrightarrow M_n \longrightarrow N_{n-1} \longrightarrow 0$$

if such an extension exists. Similarly suppose inductively that a $\mathbf{k}[\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})]$-module M_{n-1} has been constructed , $n \geq 2$, satisfying the local requirements above. Define N_n to be a nontrivial extension (if one exists)

$$0 \longrightarrow \beta \longrightarrow N_n \longrightarrow M_{n-1} \longrightarrow 0.$$

In this way we construct a family of modules $\{M_i, N_j\}$ which possibly terminates at some point. Obviously if M_t does not exist for some t then N_j will not exist for $j \geq t+1$. Hence the index where the M_i terminate and the index where the N_j terminate will differ by at most 1. We claim that these modules $\{M_i, N_j\}$ are canonically defined and up to isomorphism independent of all choices. This will follow from the following computation.

PROPOSITION 4.4: *The statements below are taken to be vacuous if a particular N_i or M_j does not exist.*

(1) $\dim_{\mathbb{F}_p} \widetilde{\operatorname{Ext}}(N_n, \alpha) \leq 1$, $\dim_{\mathbb{F}_p} \widetilde{\operatorname{Ext}}(M_n, \beta) \leq 1$.
(2) $\widetilde{\operatorname{Ext}}(N_n, \beta) = 0$, $\widetilde{\operatorname{Ext}}(M_n, \alpha) = 0$.

Proof: The long exact sequence arising from $0 \longrightarrow \beta \longrightarrow N_n \longrightarrow M_{n-1} \longrightarrow 0$ yields

$$0 \longrightarrow \operatorname{Hom}(\beta, \beta) \longrightarrow \widetilde{\operatorname{Ext}}(M_{n-1}, \beta) \longrightarrow \widetilde{\operatorname{Ext}}(N_n, \beta) \longrightarrow \widetilde{\operatorname{Ext}}(\beta, \beta) = 0.$$

As $\operatorname{Hom}(\beta, \beta) \cong \mathbb{F}_p$, we conclude that

$$\dim_{\mathbb{F}_p} \widetilde{\operatorname{Ext}}(M_{n-1}, \beta) = \dim_{\mathbb{F}_p} \widetilde{\operatorname{Ext}}(N_n, \beta) + 1. \tag{63}$$

On the other hand the short exact sequence $0 \longrightarrow \alpha \longrightarrow M_n \longrightarrow N_{n-1} \longrightarrow 0$ yields

$$0 \longrightarrow \widetilde{\operatorname{Ext}}(N_{n-1}, \beta) \longrightarrow \widetilde{\operatorname{Ext}}(M_n, \beta) \longrightarrow \widetilde{\operatorname{Ext}}(\alpha, \beta) \hookrightarrow \mathbb{F}_p.$$

Hence

$$\dim_{\mathbb{F}_p} \widetilde{\operatorname{Ext}}(M_n, \beta) \leq \dim_{\mathbb{F}_p} \widetilde{\operatorname{Ext}}(N_{n-1}, \beta) + 1. \tag{64}$$

From equations (63), (64), and the hypotheses that

$$\widetilde{\operatorname{Ext}}(N_1, \beta) = 0 \text{ and } \dim_{\mathbb{F}_p} \widetilde{\operatorname{Ext}}(M_1, \beta) \leq 1$$

we deduce that

$$\dim_{\mathbb{F}_p} \widetilde{\operatorname{Ext}}(M_n, \beta) \leq 1 \text{ and } \widetilde{\operatorname{Ext}}(N_n, \beta) = 0 \text{ for all } n.$$

The remaining pair of statements, viz.

$$\dim_{\mathbb{F}_p} \widetilde{\operatorname{Ext}}(N_n, \alpha) \leq 1 \text{ and } \widetilde{\operatorname{Ext}}(M_n, \alpha) = 0 \text{ for all } n,$$

are proved in an analogous manner. ∎

The constructed M_n and N_n are indecomposable p-torsion Gal $(\overline{\mathbb{Q}}/\mathbb{Q})$-modules with Jordan–Hölder series which may be illustrated as:

⋮
α
β
α

and

⋮
β
α
β

, respectively.

We now claim that these are all such indecomposables.

Proposition 4.5: *Suppose M is a finite p-torsion* Gal $(\overline{\mathbb{Q}}/\mathbb{Q})$*-module all of whose constituents are isomorphic to α or β. Then if M satisfies* 1), 2), *and* 3)

$$M \cong (\oplus M_i) \oplus (\oplus N_j).$$

Proof: We argue by induction on the length of M. The statement is clearly true for M of length 1—then we must have $M \equiv \alpha = M_1$ or $M \equiv \beta = N_1$ Assume the statement is true for length $n-1$. Such an M of length n must have a submodule isomorphic to α or β. We suppose we have $0 \longrightarrow \alpha \longrightarrow M \longrightarrow M' \longrightarrow 0$, the case of M containing β being analogous. If M is decomposable, then we are done by induction. Hence suppose M is indecomposable. By (2) of Proposition 4.4 we must have then $M' \cong \bigoplus_{s=1}^{r} N_{j(s)}$, which we assume ordered so that $N_{j(1)} \subseteq N_{j(2)} \subseteq \ldots \subseteq N_{j(r)}$. If $r = 1$ then again we are done since then $M \cong M_{j(1)+1}$. So we suppose $r > 1$. The class $c = (c_s)_{s=1}^{r}$ of the extension $0 \longrightarrow \alpha \longrightarrow M \longrightarrow M' \longrightarrow 0$ in $\widetilde{\mathrm{Ext}}\,(M' = \oplus N_{j(s)}, \alpha) \cong \oplus \widetilde{\mathrm{Ext}}\,(N_{j(s)}, \alpha)$ satisfies $c_s \neq 0$, $1 \le j \le r$ as M is indecomposable. For $\lambda \in \mathbb{F}_p^{\times}$ consider the map

$$i_\lambda = (1, \lambda, 0, \ldots, 0)\colon N_{j(1)} \hookrightarrow \bigoplus_{s=1}^{r} N_{j(s)}.$$

Since $\dim(\widetilde{\mathrm{Ext}}\,(N_{j(1)}, \alpha)) \le 1$ it is possible to choose λ so that $i_\lambda^*(c) = 0$. But then $N_{j(1)}$ can be split off of M. This contradiction then establishes the proposition. ∎

With these preparations we now turn our attention to the multiplicity question. Let $V := H^1_{\mathrm{par}}(Y_0(N)_{\overline{\mathbb{Q}}}, \overline{\vartheta}_p)\langle \mathbf{m} \rangle^\vee$ denote the $\mathbf{m}$-primary component of $H^1_{\mathrm{par}}(Y_0(N)_{\overline{\mathbb{Q}}}, \overline{\vartheta}_p)^\vee$. Now V is a $\mathbb{T}[\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})]$- module all of whose constituents are α or β. Hence by Proposition 4.5 $V \cong (\oplus M_i) \oplus (\oplus N_j)$. Note that the last two steps in any Jordan–Hölder filtration of V are annihilated by $\mathbf{m}$ by the

Eichler–Shimura relations. By Theorem 1.1 we know Multiplicity One for α in $V[\mathbf{m}]$ or for $\beta \in V/\mathbf{m}V$. Hence the decomposition of V into a direct sum of indecomposables must be of the form

$$V \cong A \oplus (\oplus\beta\text{'s}), \tag{65}$$

where $A \cong M_n$ or $A \cong N_n$. Therefore the dual $\mathrm{Hom}_{\mathbb{F}_p}(V, \mathbb{F}_p(k-1))$ of V is $A^* \oplus (\oplus\alpha\text{'s})$. Now $H^1_{\mathrm{par}}(Y_0(N)_{\overline{\mathbb{Q}}}, \overline{\vartheta}_p)$ is self-dual since Poincaré Duality gives a perfect pairing

$$H^1_{\mathrm{par}}(Y_0(N)_{\overline{\mathbb{Q}}}, \overline{\vartheta}_p) \times H^1_{\mathrm{par}}(Y_0(N)_{\overline{\mathbb{Q}}}, \overline{\vartheta}_p) \longrightarrow \mathbb{F}_p(1-k).$$

The Hecke algebra is self-adjoint with respect to this pairing. Moreover $\mathbb{T}_p = \mathbb{T} \otimes \mathbb{Z}_p$ is a complete semi-local ring and hence $\mathbb{T}_{\mathbf{m}}$ is a direct factor: $\mathbb{T}_p = \mathbb{T}_{\mathbf{m}} \times \mathbb{T}'_{\mathbf{m}}$. As the action of $\mathbb{T}$ is self-adjoint with respect to the auto-duality of $H^1_{\mathrm{par}}(Y_0(N)_{\overline{\mathbb{Q}}}, \overline{\vartheta}_p)$ it follows that V is self-dual.

Taking the dual reverses the steps in a Jordan–Hölder filtration, interchanging α's and β's as it goes. For example the dual of $\begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline \alpha \\ \hline \end{array}$ is

$$\begin{array}{|c|}\hline \beta \\ \hline \alpha \\ \hline \beta \\ \hline \end{array} \quad \text{and the dual of} \quad \begin{array}{|c|}\hline \beta \\ \hline \alpha \\ \hline \beta \\ \hline \alpha \\ \hline \end{array} \quad \text{is} \quad \begin{array}{|c|}\hline \beta \\ \hline \alpha \\ \hline \beta \\ \hline \alpha \\ \hline \end{array}$$

In general, the dual of M_{2n} is M_{2n} and the dual of M_{2n-1} is N_{2n-1}. Similarly the dual of N_{2n} is N_{2n} and the dual of N_{2n-1} is M_{2n-1}. Hence if $V \cong A \oplus (\oplus\beta\text{'s})$ with $A \cong M_n$ or $A \cong N_n$ and V is self-dual, then we must have $V \cong \alpha \oplus \beta$, $V \cong M_{2n}$, or $V \cong N_{2n}$.

Proposition 4.6: *The Galois module $V[\mathbf{m}] = H^1_{\mathrm{par}}(Y_0(N)_{\overline{\mathbb{Q}}}, \overline{\vartheta}_p)^\vee[\mathbf{m}]$ has $\alpha = \chi_0$ as a submodule.*

Proof: We have the diagram of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations

$$H^1_{\mathrm{par}}(Y_0(N)_{\overline{\mathbb{Q}}}, \vartheta_p) \subseteq H^1(Y_0(N)_{\overline{\mathbb{Q}}}, \vartheta_p) \longrightarrow \bigoplus_{\textit{oriented cusps}} \mathbb{Z}_p(1-k)$$

which under the étale/crystalline correspondence gives an F^{k-1}-part

$$S_k \subseteq M_k \xrightarrow{a_0} \mathbb{Z}_p[\textit{oriented cusps}].$$

Let $I = I(E) \subseteq \mathbb{T}$ be the ideal defined by an Eisenstein series E of weight k for $\Gamma_0(N)$. Then the kernel of I on M_k is $\mathbb{Z}_p \cdot E$. This means that $E \bmod a_0$ defines a class $\langle E \rangle$ in parabolic cohomology; $\langle E \rangle \in S_k/a_0 S_k = F^{k-1}$. Using the correspondence between Galois and crystalline upon reducing to $\mathbb{Q}_p^{\mathrm{unr}}$ we have the injection $\mathbb{F}_p\{k-1\} \hookrightarrow H^1_{crys}$ corresponds to $\mathbb{F}_p(1-k) \hookrightarrow H^1_{\acute{e}t}$. This injection is an injection of Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$-modules (and not just Gal$(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-modules!) because the kernel of the Eisenstein ideal is a Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$-module. In turn we obtain a surjection of Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$-modules $H^1_{\acute{e}t} \longrightarrow \mathbb{F}_p$. ∎

Since $V[\mathbf{m}]$ has α as a submodule this means $V \cong M_{2n}$ or $V \cong \alpha \oplus \beta$. If $V \cong M_{2n}$, then $V[\mathbf{m}]$ must consist of the first two steps of the filtration for any more would result in at least two α's. Hence in any case we get an exact sequence

$$0 \longrightarrow \alpha \longrightarrow V[\mathbf{m}] \longrightarrow \beta \longrightarrow 0$$

and $\dim_{\mathbf{k}} H^1_{\mathrm{par}}(Y_0(N)_{\overline{\mathbb{Q}}}, \overline{\vartheta}_p)[\mathbf{m}] = 2$. Accumulating all the hypotheses used we have then the Multiplicity One result below:

THEOREM 4.7: *Let N be a prime and $\mathbb{T}$ the weight k Hecke algebra for $\Gamma_0(N)$. Suppose $\mathbf{m} \subseteq \mathbb{T}$ is a* new *maximal ideal of residue characteristic $p > \max(3, k)$ with $\rho_{\mathbf{m}}$ reducible. Suppose that $N \not\equiv 1 \bmod p$, $U_N \not\equiv 1 \bmod \mathbf{m}$. Moreover if $2 < k < p-1$, suppose $p \nmid L(0, \omega^{k-1}\omega^{p-k})$. Then*

$$\dim_{\mathbb{F}_p} H^1_{\mathrm{par}}(Y_0(N)_{\overline{\mathbb{Q}}}, \overline{\vartheta}_p)[\mathbf{m}] = \dim_{\mathbb{F}_p} H^1_{\mathrm{par}}(Y_0(N)_{\overline{\mathbb{Q}}}, \overline{\vartheta}_p)/\mathbf{m}H^1_{\mathrm{par}}(Y_0(N)_{\mathbb{Q}}, \overline{\vartheta}_p) = 2.$$

Under these hypotheses the ring $\mathbb{T}_{\mathbf{m}}$ is Gorenstein.

5. Companion forms

We begin by recalling some results of Section 2; this will also serve to review our earlier notation. For a positive integer N denote by $\phi\colon E \longrightarrow Y_1(N)$ the universal elliptic curve. Set $\vartheta_p = \mathrm{Symm}^{k-2}(R^1\phi_*\mathbb{Z}_p)$ and $\overline{\vartheta}_p = \vartheta_p/p\vartheta_p$. Let $\mathbb{T}$ be the weight k Hecke algebra for $\Gamma_1(N)$. Suppose $\mathbf{m} \subseteq \mathbb{T}$ is a maximal ideal with residue field $\mathbf{k} = \mathbf{k}[\mathbf{m}]$ of characteristic p, $(p, N) = 1$. By the duality between the Hecke algebra and cusp forms, $\mathbf{m}$ corresponds to a normalized cusp form $f = \sum_{n=1}^{\infty} a_n q^n$ of weight k for $\Gamma_1(N)$ with coefficients in the finite field $\mathbf{k}$. We assume that the representation $\rho_{\mathbf{m}}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(2, \mathbf{k})$ is irreducible. Then by the Multiplicity One Theorem (Theorem 2.1), if $p > k$ then $V = V_{\mathbf{m}} = H^1_{\mathrm{par}}(Y_1(N)_{\overline{\mathbb{Q}}}, \vartheta_p)^{\vee}[\mathbf{m}]$ is isomorphic to the $\mathbf{k}[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module determined by

$\rho_{\mathbf{m}}$. Moreover the $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representation V is crystalline. The F-crystal $M = M_{\mathbf{m}}$ corresponding to the dual of V has a canonical filtration

$$M_{\mathbf{m}} = M = F^0 \supset F^{k-1} \supset 0 \tag{66}$$

with $F^{k-1} \cong S_k(\Gamma_1(N))[\mathbf{m}] = H^0(X_0, \omega^{\otimes k}(-\text{cusps}))[\mathbf{m}]$ and

$$F^0/F^{k-1} \cong S_k(\Gamma_1(N))^*[\mathbf{m}] = H^0(X_0, \omega^{\otimes k}(-\text{cusps}))[\mathbf{m}] = H^1(X_0, \omega^{\otimes 2-k})[\mathbf{m}]$$

by Serre Duality. There are maps $\varphi^{k-1}\colon F^{k-1} \longrightarrow M$ and $\varphi^0\colon F^0/F^{k-1} \longrightarrow M$. By definition the maximal ideal $\mathbf{m}$ in the Hecke algebra $\mathbb{T}$ is *ordinary* if $\mathrm{Tr}\,\varphi^0 \neq 0$, i.e. if and only if $\varphi^0(F^0) \not\subseteq F^{k-1}$. In this case there is an exact sequence

$$0 \longrightarrow \langle\varphi^0(F^0)\rangle \cong \mathbf{k}\{0\} \longrightarrow M_{\mathbf{m}} \longrightarrow M/\langle\varphi^0(F^0)\rangle \cong \mathbf{k}\{k-1\} \longrightarrow 0. \tag{67}$$

Hence if $\mathbf{m}$ is ordinary there is an exact sequence of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-modules

$$0 \longrightarrow \alpha\chi^{k-1} \longrightarrow V \longrightarrow \beta \longrightarrow 0, \tag{68}$$

where χ is the p-cyclotomic character and α, β are unramified. Furthermore we see that the exact sequence (67) is split if and only if $\varphi^{k-1}(F^{k-1}) \subseteq F^{k-1}$. For in this case F^{k-1} is the complement. Of course note that the sequence (68) is split if and only if the $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-module V is tamely ramified. Hence we have the following criterion:

LEMMA 5.1: *$\rho_{\mathbf{m}}$ is tamely ramified at p if and only if $\varphi^{k-1}(f) \in F^{k-1}$.*

The criterion for $\rho_{\mathbf{m}}$ to be tamely ramified at p provided by Lemma 5.1 is virtually tautological. However Serre conjectured ([16]) that whether $\rho_{\mathbf{m}}$ is tamely ramified is detected by the existence of "companion forms" to f. This conjecture was recently proven by Gross ([9]). Serre's criterion is as follows:

THEOREM 5.2: (Gross) *Suppose $p > \max(3, k)$ and $a_p \neq 0$. Then $\rho_{\mathbf{m}}$ is tamely ramified at p if and only if there is a normalized eigenform $g = \sum b_n q^n$ of weight $k' = p + 1 - k$ for $\Gamma_1(N)$ over $\mathbf{k}$ such that $n^k b_n = n a_n$ for all $n \geq 1$.*

The pair of forms (f, g) as in the above theorem are called companion forms. This relationship is equivalently expressed as $\rho_f \otimes \chi = \rho_g \otimes \chi^k$.

Gross's proof reduces the question concerning forms $\bmod p$ of weight k on $\Gamma_1(N)$ to forms of weight 2 on $\Gamma_1(Np)$. As a further illustration of crystalline techniques applied to forms of higher weight we give in this section a direct proof of Theorem 5.2. In many ways it is simpler to work with good reduction and higher weight rather than bad reduction and weight 2. We do however note that Gross's proof also yields the case $p = k$ provided $a_p^2 \neq \varepsilon(p)$ whereas crystalline methods do not directly apply here. The techniques of Abrashkin [1] might help however for a crystalline treatment of this limiting case.

One direction of Theorem 5.2 is easy. Suppose a companion form g to f exists. Then by definition there is a form g of weight $k' = p+1-k$ on $\Gamma_1(N)$ such that $\rho_f = \rho_g \otimes \chi^{k-1}$. Then as in (68) there are exact sequences

$$0 \longrightarrow \alpha\chi^{k-1} \longrightarrow V_f \longrightarrow \beta \longrightarrow 0 \qquad \alpha,\ \beta \text{ unramified,} \tag{69}$$

$$0 \longrightarrow \alpha'\chi^{k'-1} \longrightarrow V_g \longrightarrow \beta' \longrightarrow 0 \qquad \alpha',\ \beta' \text{ unramified.} \tag{70}$$

Twisting the latter sequence (70) by χ^{k-1} gives

$$0 \longrightarrow \alpha' \longrightarrow V_g \otimes \chi^{k-1} = V_f \longrightarrow \beta'\chi^{k-1} \longrightarrow 0. \tag{71}$$

Since $p > k \geq 2$ $\alpha' \neq \alpha\chi^{k-1}$. Hence $\alpha = \beta'$, $\beta = \alpha'$, and $V_f = \alpha\chi^{k-1} \oplus \beta$.

The other direction of Theorem 5.2 is the substantial implication. In light of Lemma 5.1, this is the assertion that if $\varphi^{k-1}(f) \in F^{k-1}$ then a companion form to f exists. We shall compute $\varphi^{k-1}(f) \bmod F^{k-1}$. Let $X = X_1(N)$, $(p,N) = 1$. On the formal completion $\hat{X}^{\mathrm{ord}}$ there is a Frobenius lift Φ^{ord} induced by taking an elliptic curve E to E modulo its μ_p-subgroup. This Φ^{ord} operates on Ω_X, ω, $\overline{\vartheta}_p$, etc. respecting filtration. On ω, Φ^{ord}/p: $\omega^p \longrightarrow \omega$ is given by $1/e_{p-1}$ where e_{p-1} is the normalized Eisenstein series of weight $p-1$ taken modulo p.

Let $\Sigma \subset X$ denote the supersingular locus. For $x \in \Sigma$, e_{p-1} has a simple zero at x, so $u = de_{p-1}(x) \in \omega^{p-1} \otimes \Omega_X(x) = \omega^{p+1}(x)$ makes sense. Such a u also admits an alternate description. Let E be a supersingular elliptic curve over a perfect field of characteristic p. Then Frob_p acting on $H^1_{\mathrm{crys}}(E)$ induces an isomorphism $H^1(E, \mathcal{O}_E)^p \cong \omega^{-p} \xrightarrow{\cong} \Gamma(E, \Omega_E) = \omega$. This isomorphism is multiplication by u, at least up to a factor independent of E. Let Φ be a local Frobenius lift near a supersingular point. The assertion follows from the relation in characteristic p, $\nabla \circ \Phi^* = d\Phi^* \circ \nabla = 0$.

For applying this to a class z in $H^1_{\mathrm{DR}}(E)$ which locally generates $H^1(E, \mathcal{O}_E)$:

$$\Phi^* z = e_{p-1} \cdot z + \omega(z), \quad \text{for } \omega \in \Gamma(E, \Omega_E^1),$$

and hence $\nabla\Phi^* z = de_{p-1} \cdot z + \nabla\omega(z)$. Calculating modulo $\Gamma(E, \Omega^1_E)$, we have $\nabla\omega(z) = (\text{Kodaira–Spencer}) \times \omega$, and hence $\omega(z) \equiv -de_{p-1}(z)$ using Kodaira–Spencer.

For a more precise notation let Φ^{ss} be a Frobenius lift near Σ on $\hat{X}$. Then

$$(\Phi^{ss} - \Phi^{ord})(z) = p(\partial z) \bmod p^2,$$

where ∂ is a Frobenius-linear derivation on an open subset contained in the ordinary locus of $X_0 = X \times \mathbb{F}_p$. We want to determine its poles along Σ. If $x \in \Sigma$ and z is a local coordinate near x then $\frac{(\Phi^{ss})^*}{p}(dz)$ is regular at x. But

$$\frac{(\Phi^{ord})^*}{p}(dz) \equiv \frac{(dz)^p}{e_{p-1}^2} \equiv \frac{(dz)^p}{u^2z^2dz^{p-1}} \equiv \frac{dz}{u^2z^2} + \text{ (lower order terms) } \bmod p;$$

here u is the leading term of e_{p-1}. We shall see that $\Phi^{ord}(z) \bmod p^2$ has at most simple poles, so it follows that $d(\partial z) \equiv -\frac{dz}{u^2z^2} \bmod (p, \text{regular})$ and therefore $\partial z = \frac{1}{u^2z} + \text{ (regular)}$.

We now check that ∂ has at most simple poles, i.e., that $\Phi^{ord}(z) \bmod p^2$ has at most a simple pole. For this let w be a local coordinate at Frob(x). Then locally $X_0(p) \subset X \times X$ is defined by an equation $g(z, w)$ with

$$g(z, w) = (z^p - w)(z - w^p) \bmod p.$$

Let $\text{pr}_1, \text{pr}_2\colon X \times X \longrightarrow X$ denote projection onto the first and second factors, respectively. The μ_p-type subgroup defines a rational section of pr_1 and Φ^{ord} is given by composing with the second projection. In other words in local coordinates we have $g(z, \Phi^{ord}(z)) = 0$ and $\Phi^{ord}(z) \equiv z^p \bmod p$, so

$$g(z, \Phi^{ord}(z)) = (\Phi^{ord}(z) - z^p)(z - \Phi^{ord}(z)^p) + \text{ (terms regular in } z\text{) } \bmod p^2.$$

Therefore $\frac{\Phi^{ord}(z) - z^p}{p}(z - z^{p^2})$ is regular $\bmod p$.

Finally we consider $f \in \Gamma(X_0, \omega^k) \cong \Gamma(X_0, \omega^{k-2} \otimes \Omega^1(\infty))$. Recall that $\mathcal{E} = R^1\pi_{*,\text{crys}}(\mathcal{O})$. The element f defines a class in $H^1_{\text{crys}}(Y_0, \text{Symm}^{k-2}(\mathcal{E}))$, which is in F^{k-1}. We want to compute $\varphi^{k-1}(f) \bmod F^1$, which is in $H^1(X_0, \omega^{2-k})$. It suffices by Serre Duality to compute the product $\langle \varphi^{k-1}(f), \beta \rangle$ for $\beta \in \Gamma(X_0, \omega^{k-2} \otimes \Omega)$, i.e., β a cusp form. We need explicit computations in the de Rham complex $\text{Symm}^{k-2}(\mathcal{E}) \xrightarrow{\nabla} \text{Symm}^{k-2}(\mathcal{E}) \otimes \Omega^1(\infty)$ in order to give a represention of $\varphi^{k-1}(f)$. Firstly note the formulas for a function h:

$$h(z + t) - h(z) = \sum_{n=1}^{\infty} \frac{\partial^n h}{\partial z^n}(z) \frac{t^n}{n!}, \tag{72}$$

$$h(z+t)d(z+t) - h(z)dz = d\Big(\sum_{n=1}^{\infty} \frac{\partial^{n-1}h}{\partial z^{n-1}}(z)\frac{t^n}{n!}\Big), \tag{73}$$

which it suffices to check for $h(z) = z^m$. The same formulas hold for sections of a bundle with integrable connection, replacing $\partial/\partial z$ by $\nabla(\partial/\partial z)$. Let us consider this in some generality.

Suppose V is $\mathbb{Z}_p$ or an unramified extension, $C = U_1 \cup U_2$ is an open affine cover, and E is a Frobenius-crystal on C. This means that there is a connection $\nabla\colon E \longrightarrow E \otimes \Omega^1_{C/V}$, which is integrable as the dimension is 1. Also, if Φ_i is a Frobenius-lift on the formal scheme $\hat{U}_i$ (i=1,2), then we have Φ_i-linear $\varphi_i\colon E|\hat{U}_i \longrightarrow E|\hat{U}_i$ which is parallel. Set $U_{1,2} = U_1 \cap U_2$ and let D be the divided power hull for $U_{1,2} \hookrightarrow U_1 \times_V U_2$. We can define crystalline cohomology either by the total complex associated to

$$\Big(E(\hat{U}_1) \otimes \Omega^\bullet_{\hat{U}_1}\Big) \oplus \Big(E(\hat{U}_2) \otimes \Omega^\bullet_{\hat{U}_2}\Big) \longrightarrow E(\hat{U}_{1,2}) \otimes \Omega^\bullet_{\hat{U}_{1,2}} \tag{74}$$

or by the total complex associated to

$$\Big(E(\hat{U}_1) \otimes \Omega^\bullet_{\hat{U}_1}\Big) \oplus \Big(E(\hat{U}_2) \otimes \Omega^\bullet_{\hat{U}_2}\Big) \longrightarrow E(D(\widehat{U_1 \times_V U_2})) \otimes \Omega^\bullet_{\hat{U}_1 \times_V U_2}. \tag{75}$$

The complex (75) maps to the complex (74). The first complex (74) is smaller, while the second (75) has Frobenius action (φ_1, φ_2, $\varphi_1 \times_V \varphi_2$). If $\alpha \in \Gamma(C, E \otimes \Omega^1)$ is a 1-form, we get a 1-cocycle in the first complex (74) with components $(\alpha|\hat{U}_1, \alpha|\hat{U}_2, 0)$, the 0 being in the $(1,2)$-component. Let z be a local coordinate on $U_{1,2}$, $z_i = \mathrm{pr}_i^*(z)$ which is then a function on $U_1 \times_V U_2$, and $\partial_z = \partial/\partial z$. Lift to a 1-cocycle in the second complex (75) as follows:

$$\left(\alpha|\hat{U}_1,\ \alpha|\hat{U}_2,\ \sum_{n=0}^{\infty} \mathrm{pr}_2^*(\nabla(\partial_z)^n i(\partial_z)\alpha)\frac{(z_1-z_2)^{n+1}}{(n+1)!}\right), \tag{76}$$

where the interior multiplication (i.e., contraction) $i(\partial_z)\alpha$ is sometimes also denoted $\langle \partial_z, \alpha\rangle$. This is closed since

$$\mathrm{pr}_1^*(\alpha) - \mathrm{pr}_2^*(\alpha) = \nabla\left(\sum_{n=1}^{\infty} pr_2^*(\nabla(\partial_z)^{n-1} i(\partial_z)\alpha)\frac{(z_1-z_2)^n}{n!}\right) \tag{77}$$

on $D(U_1 \times_V U_2)$. Now apply Frobenius to all components to deduce that $\mathrm{Frob}^*(\alpha)$ is given by the 1-cocycle:

$$\left(\varphi_1^*(\alpha|\hat{U}_1),\ \varphi_2^*(\alpha|\hat{U}_1),\ \sum_{n=0}^{\infty} pr_2^*\varphi_2^*(\nabla(\partial_z)^n i(\partial_z)\alpha)\frac{(\Phi_1(z_1)-\Phi_2(z_2))^{n+1}}{(n+1)!}\right). \tag{78}$$

If we map to the first complex and divide by $F^1(H^1_{\mathrm{DR}}(E)) = \mathrm{Im}\,(\Gamma(C, E \otimes \Omega^1))$, i.e., map to $H^1(C, E)$ (usual cohomology), only the third component survives and gives a Cech 1-cocycle.

Now in our case α is a section f of $\omega^{\otimes k} = \omega^{\otimes k-2} \otimes \Omega^1 \subseteq \mathrm{Symm}^{\,k-2}(E) \otimes \Omega^1$. From equation (77) we get

$$(79) \quad \varphi_1^*(f) - \varphi_2^*(f) = \nabla\left(\sum_{n=1}^{\infty} \varphi_2^*(\nabla(\partial_z)^{n-1} i(\partial_z) f) \frac{(\Phi_1(z) - \Phi_2(z))^n}{n!}\right).$$

We need in fact Frob^*/p^{k-1}. Note that Frob^* on $\nabla(\partial_z)^{n-1} f$ is only divisible by $p^{k-2-(n-1)} = p^{k-n-1}$ if $n \leq k-1$, respectively p^0 if $n \geq k-1$. But $(\varphi_1(z) - \varphi_2(z))^n$ is divisible by p^n, and for $n!$ we need a p only if $n \geq p$, etc. So the nth term in $\dfrac{\varphi_1^* - \varphi_2^*}{p^{k-1}}$ has p-exponent

$$(80) \qquad -(k-1) + \max(k-n-1, 0) + n - \sum_{\nu=1}^{\infty} \left[\frac{n}{p^\nu}\right].$$

For $n \leq k-1$ this is 0. For $n > k-1$ this is

$$n + 1 - k - \sum_{\nu=1}^{\infty} \left[\frac{n}{p^\nu}\right] > n + 1 - k - \frac{n}{p-1}.$$

As $k < p$, the minimum is obtained at $n = k$ where we obtain $1 - \frac{k}{p-1} \geq 0$. Hence these terms are all congruent to 0 $\mathrm{mod}\, p$. Finally φ_2^* respects filtration, so $\mathrm{mod}\, F^1$ only the term $\nabla(\partial_z)^{k-2} i(\partial_z) f$ survives. Also modulo F^1 this is (up to a factor) given by multiplication using the Kodaira–Spencer class—see the beginning of Section 1.

Now the product $\langle \varphi^{k-1}(f), \beta \rangle$ is equal to the sum of the residues at Σ, after multiplication with β; see [8]. By the black magic of crystalline cohomology (cf. equation (76)) we have have to pair the term inside $\nabla(\ )$ with β and compute residues. We only need its value modulo F^1. Hence finally

$$(81) \qquad \langle \varphi^{k-1}(f), \beta \rangle = \sum_{x \in \Sigma} \frac{f^p \beta}{u^k}(x),$$

where again u is the leading term of e_{p-1}.

Now $\varphi^{k-1}(f) \in F^1$ if and only if $\langle \varphi^{k-1}(f), \beta \rangle = 0$ for all cusp forms β. By Equation 81 this occurs exactly when

$$\sum_{x \in \Sigma} \mathrm{Res}_x\left(\frac{f^p \beta}{u^{k-1} e_{p-1}}\right) = 0 \quad \text{for all } \beta.$$

But using the Residue Theorem and Serre Duality this happens if and only if there exists $\gamma \in \Gamma_{\mathrm{mero}}(X_0, \omega^{2-k})$ such that γ has a simple pole at $x \in \Sigma$ with residue equal to that of $f^p/u^{k-1}e_{p-1}$ and no other singularity. Multiplying by e_{p-1} we see that this is equivalent to the claim that there exists $g \in \Gamma(X_0, \omega^{p+1-k})$ which at Σ has the same values as f^p/u^{k-1}. The values of g on Σ uniquely determine it; otherwise for two such their difference would be divisible by the Eisenstein series and so have negative degree. The Hecke eigenvalues of g for T_ℓ, $\ell \neq p$, are uniquely determined. Namely if f has eigenvalues a_ℓ, $\ell \neq p$ then a priori by Hecke equivariance g has eigenvalues a_ℓ^p/ℓ^{k-1}. This also holds for T_p by the crystalline theory. Thus ρ_g is determined. If we let σ denote Frobenius, then in fact $\rho_g = \rho_{f^\sigma \otimes \chi^{-(k-1)}}$. This shows that ρ_g is irreducible and therefore g is a cusp form. So twisting by the inverse of Frobenius g determines a companion form to f, concluding the proof.

References

[1] V. Abrashkin, *Ramification in étale cohomology*, Inventiones Mathematicae **101** (1990), 631–640.

[2] H. Carayol, *Sur les représentations galoisiennes* modulo ℓ *attachées aux formes modulaires*, Duke Mathematical Journal **59** (1989), 785–801.

[3] R. Coleman and J. Voloch, *Companion forms and Kodaira-Spencer theory*, Inventiones Mathematicae **110** (1992), 263–281.

[4] H. Davenport (revised by H. Montgomery), *Multiplicative Number Theory*, Springer, Berlin–Heidelberg–New York, 1980.

[5] P. Deligne, *Formes modulaires et représentations ℓ-adiques*, Exposé 355, Séminaire N. Bourbaki 1968/1969. Lecture Notes in Mathematics 179, Springer, Berlin–Heidelberg–New York, 1969.

[6] P. Deligne and N. Katz, *SGA 7 II* (Lecture Notes in Mathematics 340), Springer, Berlin–Heidelberg–New York, 1973.

[7] G. Faltings, *Crystalline cohomology and Galois representations*, in *Algebraic Analysis, Geometry, and Number Theory*, Proceedings of the JAMI Inaugural Conference (J. Igusa, ed.), Johns Hopkins Univ. Press, 1989

[8] G. Faltings, *Hodge–Tate structures and modular forms*, Annals of Mathematics **278** (1987), 133–149.

[9] B. Gross, *A tameness criterion for Galois representations associated to modular forms (mod p)*, Duke Mathematical Journal **61** (1990), 445–517.

[10] A. Grothendieck, *SGA7 I, Exposé IX* (Lecture Notes in Mathematics 288, pp. 313–523), Springer, Berlin–Heidelberg–New York, 1972.

[11] N. Katz and B. Mazur, *Arithmetic Moduli of Elliptic Curves,* Annals of Mathematics Studies **108**, Princeton University, 1985.

[12] M. Kurihara, *Ideal class groups of cyclotomic fields and modular forms of level* 1, Journal of Number Theory **45** (1993), 281–294.

[13] R. Livné, *On the conductors of* mod ℓ *Galois representations coming from modular forms,* Journal of Number Theory **31** (1989), 133–141.

[14] B. Mazur, *Modular curves and the Eisenstein ideal,* Publications Mathématiques de l'Institut des Hautes Études Scientifiques **47** (1978), 33–186.

[15] K. Ribet, *On modular representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *arising from modular forms,* Inventiones Mathematicae **100** (1990), 431–476.

[16] J.-P. Serre, *Letter to J.-M. Fontaine,* May 27, 1979.

[17] J.-P. Serre, *Sur les représentations modulaires de degré* 2 *de* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Duke Mathematical Journal **54** (1987), 179–230.

[18] D. Ulmer, *L-functions of universal elliptic curves over Igusa curves,* American Journal of Mathematics **112** (1990), 687–712.

[19] J. Weisinger, *Some results on classical Eisenstein series and modular forms over function fields,* Thesis, Harvard University, 1977.